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G. R. Goldsbrough

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THE THEORY OF THE DIVISIONS IN SATURN'S RINGS

By G. R. GOLDSBROUGH, F.R.S.

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In an earlier paper on this subject the author (1921) proposed a theory of the rings which showed satisfactory agreement with the observed measurements of the rings. The mathematical method was, however, subjected to criticism. In the present paper the subject is again attacked by an entirely different method which is free from the objections raised against the first method.

A family of periodic orbits of the particles forming the ring, when perturbed by a satellite, is constructed, and the stability of these orbits is examined by the method of small displacements. Stability determined in this way is shown to have a real meaning when applied to the problem in hand.

The positions of instability of the particles lead to the divisions in the ring and the inner and outer boundaries, close agreement with observation being obtained. The analysis, though quite different from that of the earlier paper, reproduces its main features, and introduces further points of interest.

In an earlier paper the author (1921) endeavoured to show that the principal features of Saturn's rings could be accounted for by the perturbing effects of the satellites, principally Mimas. The theory so formulated showed a good agreement with observation. The mathematical method adopted was, however, criticized by Greaves (1922), Brown (1924) and Pendse (1935). The first two raised objections to the use of the method of small motions, and the last showed the inapplicability of one of the hypotheses. It should be noted, however, that this hypothesis was not essential to the main results.

It seemed worth while therefore, in view of the interest of the problem, to re-examine it from a fresh point of view, and that is done in the present paper.

The method now used is that of periodic orbits. This method, as formulated by Poincaré, has the advantage of a foreknowledge of the precise conditions of convergence of the series involved. If the conditions are fulfilled, the series are valid for all time. Saturn being taken as the primary, a satellite is assumed to describe about it an unperturbed circular orbit as in the restricted problem of three bodies. In the plane of this orbit is a ring of equal small particles with its centre at the centre of Saturn. The mean positions of the particles are equally spaced on the circle. For this system a family of periodic orbits is determined depending upon the masses, the number of particles in the ring, and a single parameter Ω.

The particles are then given small arbitrary displacements and the equations of the variations formed. These equations can be integrated as power series and the characteristic exponents found. For motion with real characteristic exponents we use the term 'unstable', and for motion with purely imaginary exponents the term 'stable'. The question may be raised here as to whether stability or instability determined in this way has any real physical meaning. In the case of the problem in hand it can be claimed that it has. Although throughout the work only a single ring of particles is mentioned, this is an approximation to the complete problem required by the exigencies of analysis. In the more complete form of the problem a series of concentric rings of particles should be considered. In such a case the amplitudes of the vibrations of the particles of any one ring about their mean positions would of necessity be restricted if collisions with those in adjacent rings were to be avoided. If collisions did occur there would result a loss of energy and a movement to entirely new positions. Hence, if the motions of the particles are stable, in the sense mentioned above, the amplitudes of the vibrations will be restricted and no collisions will arise. But if the motions are unstable, even if the apparent instability is a faulty representation of what is really a stable motion of large amplitude, then the aforesaid collisions would still occur. So that, however it may apply in other problems, the determination of stability by small arbitrary displacements has, in the problem of Saturn's rings, a definite meaning.

The observed gaps in the rings of Saturn are found to correspond closely to just such unstable motions of the particles.

It will be seen that, although the analysis of this paper is quite different from that of the former paper, the results of that paper are almost exactly reproduced. The present work goes further, however, and shows immediately the presence of Encke's division, which required an additional hypothesis on the former occasion.

The general equations

Assume that a satellite of mass m' is moving about the planet Saturn, whose mass is M, in an unperturbed circle of radius a' with angular velocity ω', such that $\omega'^2 a'^3 = M+m$. Consider a ring of p particles all of small mass m moving in the plane of the satellite orbit. Taking an origin at the centre of Saturn and an axis through the satellite and moving with it, let the co-ordinates of the particles in polar co-ordinates referred to the moving axis be $(r_\lambda, \theta_\lambda)$, where $\lambda = 1, 2, 3, \ldots, p$.

Then, putting
$$\Theta_\lambda = \Sigma' \frac{m}{D_{\lambda\mu}} - \Sigma' \frac{m r_\lambda}{r_\mu^2} \cos(\theta_\lambda \sim \theta_\mu),$$

$$\Phi_\lambda = \frac{1}{\Delta_\lambda} - \frac{r_\lambda}{a'^2} \cos\theta_\lambda,$$

$$D_{\lambda\mu}^2 = r_\lambda^2 + r_\mu^2 - 2 r_\lambda r_\mu \cos(\theta_\lambda \sim \theta_\mu).$$

$$\Delta_\lambda^2 = r_\lambda^2 + a'^2 - 2 r_\lambda a' \cos\theta_\lambda,$$

the equations of motion become

$$\left.\begin{aligned}\frac{d^2r_\lambda}{dt^2}-r_\lambda\left(\frac{d\theta_\lambda}{dt}+\omega'\right)^2+\frac{M+m}{r_\lambda^2}&=\frac{\partial\Theta_\lambda}{\partial r_\lambda}+m'\frac{\partial\Phi_\lambda}{\partial r_\lambda},\\ \frac{d}{dt}\left\{r_\lambda^2\left(\frac{d\theta_\lambda}{dt}+\omega'\right)\right\}&=\frac{\partial\Theta_\lambda}{\partial\theta_\lambda}+m'\frac{\partial\Phi_\lambda}{\partial\theta_\lambda},\end{aligned}\right\}\tag{1}$$

$$(\lambda,\mu=1,2,3,\ldots,p.)$$

In the formulae Σ' indicates summation over all values of μ from 1 to p, except $\mu=\lambda$.

In (1) we have $2p$ differential equations of the second order to solve for the motion of the p particles. The problem is thus analogous to the restricted problem of three bodies in that one of the bodies, the satellite, is assumed to be unperturbed by the p particles, and to maintain a circular orbit about the primary.

When $m'=0$, equations (1) become Maxwell's equations (referred to moving axes) for the motion of the particles forming Saturn's ring. They have the particular solution

$$\left.\begin{aligned}r_\lambda&=a,\\ \theta_\lambda&=(\omega-\omega')\,t+2\pi\lambda/p,\\ \omega^2a^3&=M+m+\tfrac{1}{4}m\sum_{n=1}^{p-1}\operatorname{cosec}n\pi/p.\end{aligned}\right\}\tag{2}$$

The particles are now in relative equilibrium at the corners of a regular p-gon which rotates relative to the moving axes with angular velocity $(\omega-\omega')$.

We shall assume that M is large compared with m or m', and that therefore $M+m$ and $M+m'$ may be replaced by M with sufficient approximation.

When $m'\neq 0$ put $\quad r_\lambda=a(1+\rho_\lambda),\quad \theta_\lambda=(\omega-\omega')\,t+2\pi\lambda/p+\sigma_\lambda.$

Then, since Θ_λ is independent of m', and Θ_λ, Φ_λ can be expanded as power series in ρ_μ, σ_μ, which are convergent when

$$|\rho_\mu|\leqslant\bar\rho_\mu,\quad |\sigma_\mu|\leqslant\bar\sigma_\mu,\quad 0\leqslant t\leqslant 2\pi/(\omega-\omega'),$$

where $\bar\rho_\mu$, $\bar\sigma_\mu$ are appropriate non-zero constants, it follows (Poincaré 1892; Moulton 1920) that the equations (1) can be solved as power series in m', convergent in the interval $0\leqslant t\leqslant 2\pi/(\omega-\omega')$, provided m' is sufficiently small. Further, if the constants of integration can be adjusted so that the solution is periodic with the period $2\pi/(\omega-\omega')$, then this solution will be valid for all time. We proceed to construct such periodic solutions, on the assumption that the mass m' of the satellite is sufficiently small to ensure the convergence.

Put $m'=\epsilon$, and assume $\quad \rho_\lambda=\sum_{n=1}^{\infty}\epsilon^n\rho_\lambda^{(n)},\quad \sigma_\lambda=\sum_{n=1}^{\infty}\epsilon^n\sigma_\lambda^{(n)}.$

Change the independent variable t to τ, $=(\omega-\omega')\,t$, and put

$$\Omega=\frac{\omega}{\omega-\omega'},\quad \frac{1}{4}\frac{m}{a^3}\sum_{n=1}^{p-1}\operatorname{cosec}n\pi/p=K.$$

Also let the derivatives with regard to τ now be denoted by dots. The left members of (1) now become, respectively,

$$a(\omega-\omega')^2\Big[\epsilon\Big\{\ddot{\rho}_\lambda^{(1)}-3\Omega^2\rho_\lambda^{(1)}-2\Omega\dot{\sigma}_\lambda^{(1)}+2K\frac{\Omega^2}{\omega^2}\rho_\lambda^{(1)}\Big\}$$

$$+\epsilon^2\Big\{\ddot{\rho}_\lambda^{(2)}-3\Omega^2\rho_\lambda^{(2)}-2\Omega\dot{\sigma}_\lambda^{(2)}+2K\frac{\Omega^2}{\omega^2}\rho_\lambda^{(2)}$$

$$-2\Omega\dot{\sigma}_\lambda^{(1)}\rho_\lambda^{(1)}-(\dot{\sigma}_\lambda^{(1)})^2+3\Omega^2(\rho_\lambda^{(1)})^2-3K\frac{\Omega^2}{\omega^2}(\rho_\lambda^{(1)})^2\Big\}$$

$$+\epsilon^3\{\ldots\}+\ldots\Big], \tag{3}$$

and
$$a^2(\omega-\omega')^2[\epsilon\{\ddot{\sigma}_\lambda^{(1)}+2\Omega\dot{\rho}_\lambda^{(1)}\}$$

$$+\epsilon^2\{\ddot{\sigma}_\lambda^{(2)}+2\Omega\dot{\rho}_\lambda^{(2)}+2\Omega\rho_\lambda^{(1)}\dot{\rho}_\lambda^{(1)}+2\dot{\rho}_\lambda^{(1)}\dot{\sigma}_\lambda^{(1)}+2\ddot{\sigma}_\lambda^{(1)}\rho_\lambda^{(1)}\}$$

$$+\epsilon^3\{\ldots\}+\ldots]. \tag{4}$$

For the right members of equations (1) we have, respectively,

$$\frac{\partial\Theta_\lambda}{\partial r_\lambda}=\left(\frac{\partial\Theta_\lambda}{\partial r_\lambda}\right)_0+\sum_\mu\left(\frac{\partial^2\Theta_\lambda}{\partial r_\lambda\,\partial r_\mu}\right)_0 a(\epsilon\rho_\mu^{(1)}+\epsilon^2\rho_\mu^{(2)}+\ldots)$$

$$+\sum_\mu\left(\frac{\partial^2\Theta_\lambda}{\partial r_\lambda\,\partial\theta_\mu}\right)_0(\epsilon\sigma_\mu^{(1)}+\epsilon^2\sigma_\mu^{(2)}+\ldots)$$

$$+\ldots, \tag{5}$$

$$\frac{\partial\Theta_\lambda}{\partial\theta_\lambda}=\left(\frac{\partial\Theta_\lambda}{\partial\theta_\lambda}\right)_0+\sum_\mu\left(\frac{\partial^2\Theta_\lambda}{\partial\theta_\lambda\,\partial r_\mu}\right)_0 a(\epsilon\rho_\mu^{(1)}+\epsilon^2\rho_\mu^{(2)}+\ldots)$$

$$+\sum_\mu\left(\frac{\partial^2\Theta_\lambda}{\partial\theta_\lambda\,\partial\theta_\mu}\right)_0(\epsilon\sigma_\mu^{(1)}+\epsilon^2\sigma_\mu^{(2)}+\ldots)$$

$$+\ldots, \tag{6}$$

$$\frac{\partial\Phi_\lambda}{\partial r_\lambda}=\left(\frac{\partial\Phi_\lambda}{\partial r_\lambda}\right)_0+\left(\frac{\partial^2\Phi_\lambda}{\partial r_\lambda^2}\right)_0 a(\epsilon\rho_\lambda^{(1)}+\epsilon^2\rho_\lambda^{(2)}+\ldots)$$

$$+\left(\frac{\partial^2\Phi_\lambda}{\partial r_\lambda\,\partial\theta_\lambda}\right)_0(\epsilon\sigma_\lambda^{(1)}+\epsilon^2\sigma_\lambda^{(2)}+\ldots)$$

$$+\ldots, \tag{7}$$

$$\frac{\partial\Phi_\lambda}{\partial\theta_\lambda}=\left(\frac{\partial\Phi_\lambda}{\partial\theta_\lambda}\right)_0+\left(\frac{\partial^2\Phi_\lambda}{\partial\theta_\lambda\,\partial r_\lambda}\right)_0 a(\epsilon\rho_\lambda^{(1)}+\epsilon^2\rho_\lambda^{(2)}+\ldots)$$

$$+\left(\frac{\partial^2\Phi_\lambda}{\partial\theta_\lambda^2}\right)_0(\epsilon\sigma_\lambda^{(1)}+\epsilon^2\sigma_\lambda^{(2)}+\ldots)$$

$$+\ldots. \tag{8}$$

In these formulae the zero derivatives are found by substituting $r_\lambda = a$, $\theta_\lambda = \tau + 2\pi\lambda/p$ after differentiating.

The method of solving the fundamental equations is to equate to zero the coefficients of the various powers of ϵ and solve the resulting equations in sequence.

Evaluation of the derivatives

The zero derivatives of the functions Θ, Φ are now required. Only a summary of the results will be given as the details may be found in the references quoted:

$$\sum_\mu \left(\frac{\partial^2\Theta_\lambda}{\partial r_\lambda \partial r_\mu}\right)_0 a\rho_\mu = -\frac{m}{a^2}\rho_\lambda \sum_{n=1}^{p-1} \left(\tfrac{1}{8}\operatorname{cosec}^3 n\pi/p - \tfrac{3}{8}\operatorname{cosec} n\pi/p\right)$$

$$+\frac{m}{a^2}\sum_\mu{}' \rho_\mu \left\{\frac{1+\sin^2(\mu-\lambda)\pi/p}{8\sin^3(\mu\sim\lambda)\pi/p} + 2\cos(\mu-\lambda)2\pi/p\right\},$$

$$\sum_\mu \left(\frac{\partial^2\Theta_\lambda}{\partial r_\lambda \partial \theta_\mu}\right)_0 \sigma_\mu = \frac{m}{a^2}\sum_\mu{}' \sigma_\mu\{\tfrac{1}{16}\operatorname{cosec}^3(\mu\sim\lambda)\pi/p + 1\}\sin(\mu-\lambda)2\pi/p,$$

$$\sum_\mu \left(\frac{\partial^2\Theta_\lambda}{\partial \theta_\lambda \partial r_\mu}\right) a\rho_\mu = \frac{m}{a^2}\sum_\mu{}' \rho_\mu\{2-\tfrac{1}{16}\operatorname{cosec}^3(\mu\sim\lambda)\pi/p\}\sin(\mu-\lambda)2\pi/p,$$

$$\sum_\mu \left(\frac{\partial^2\Theta_\lambda}{\partial \theta_\lambda \partial \theta_\mu}\right) \sigma_\mu = \frac{m}{a}\sigma_\lambda \sum_{n=1}^{p-1}\left\{\frac{1+\cos^2 n\pi/p}{8\sin^3(\mu\sim\lambda)\pi/p} + \cos 2n\pi/p\right\}$$

$$-\frac{m}{a}\sum_\mu{}' \sigma_\mu \left\{\frac{1+\cos^2(\mu-\lambda)\pi/p}{8\sin^3(\mu\sim\lambda)\pi/p} + \cos(\mu-\lambda)\pi/p\right\}.$$

The higher derivatives follow similarly.

We have also
$$\Phi_\lambda = \frac{1}{a'}\left(1+\frac{r_\lambda^2}{a'^2} - \frac{2r_\lambda}{a'}\cos\theta_\lambda\right)^{-\frac{1}{2}} - \frac{r_\lambda}{a'^2}\cos\theta_\lambda.$$

This function and its derivatives can be expanded in a series of cosines or sines of multiples of θ_λ by well-known methods. Putting $\alpha = a/a'$, we have

$$(\Phi_\lambda)_0 = \frac{1}{a'}(1+\alpha^2 - 2\alpha\cos\theta_\lambda)^{-\frac{1}{2}} - \frac{1}{a'}\alpha\cos\theta_\lambda$$

$$= \frac{1}{a'}\left[\tfrac{1}{2}b_0 + (b_1-\alpha)\cos\theta_\lambda + \sum_{n=2}^{\infty} b_n \cos n\theta_\lambda\right],$$

where $b_0, b_1, \ldots$ are the Laplace coefficients. Hence

$$\left(\frac{\partial\Phi_\lambda}{\partial r_\lambda}\right)_0 = \frac{1}{a'^2}\sum_n B_n' \cos n(\tau + 2\pi\lambda/p),$$

$$\left(\frac{\partial\Phi_\lambda}{\partial \theta_\lambda}\right)_0 = -\frac{1}{a'}\sum_n nB_n \sin n(\tau + 2\pi\lambda/p),$$

and so on, where $B_0 = \frac{1}{2}b_0$, $B_1 = b_1 - \alpha$, $B_n = b_n$, $n > 1$. The dashes attached to the B_n indicate differentiation with regard to α. It is to be noted that the coefficients B_n are independent of λ and positive.

We also require for later parts of the work the following notation and results:

$$N = 2 - \frac{1}{8}\sum_{n=1}^{p-1}\{\operatorname{cosec}^3 n\pi/p + \operatorname{cosec} n\pi/p\},$$

$$P_s = \sum_{n=1}^{p-1}\{\tfrac{1}{8}\operatorname{cosec}^3 n\pi/p + \tfrac{1}{8}\operatorname{cosec} n\pi/p + 2\cos 2n\pi/p\}\cos 2\pi ns/p,$$

$$Q_s = \sum_{n=1}^{p-1}\{\tfrac{1}{16}\operatorname{cosec}^3 n\pi/p + 1\}\sin 2\pi n/p \sin 2\pi ns/p,$$

$$R_s = \sum_{n=1}^{p-1}\{2 - \tfrac{1}{16}\operatorname{cosec}^3 n\pi/p\}\sin 2\pi n/p \sin 2\pi ns/p,$$

$$T_s = 2\sum_{n=1}^{p-1}\left\{\frac{1+\cos^2 n\pi/p}{8\sin^3 n\pi/p} + \cos 2n\pi/p\right\}\sin^2 ns\pi/p,$$

$$S_s = \frac{3}{16}\sum_{n=1}^{p-1}\frac{\cos n\pi/p}{\sin^4 n\pi/p}\sin 2sn\pi/p.$$

As p is a large integer, we may neglect $\Sigma \operatorname{cosec} n\pi/p$ compared with $\Sigma \operatorname{cosec}^3 n\pi/p$. We have then

$$T_s = -2(P_s + N) = \frac{1}{2}\sum_{n=1}^{p-1}\frac{\sin^2 ns\pi/p}{\sin^3 n\pi/p},$$

also

$$R_s = -Q_s.$$

It is to be noticed that T_s is always positive.

Integration of the equations

Taking the terms factored by ϵ, we have from the previous results

$$\left.\begin{aligned}
&\ddot{\rho}_\lambda^{(1)} - 2\Omega\dot{\sigma}_\lambda^{(1)} - 3\Omega^2\rho_\lambda^{(1)} + \frac{2K}{(\omega-\omega')^2}\rho_\lambda^{(1)} \\
&\qquad = \frac{1}{a(\omega-\omega')^2}\left[\sum_\mu\left(\frac{\partial^2\Theta_\lambda}{\partial r_\lambda \partial r_\mu}\right)_0 a\rho_\mu^{(1)} + \sum_\mu\left(\frac{\partial^2\Theta_\lambda}{\partial r_\lambda \partial \theta_\mu}\right)_0 \sigma_\mu^{(1)}\right] \\
&\qquad\qquad + \frac{1}{2aa'^2(\omega-\omega')^2}\sum_n B_n'(e^{in(\tau+2\pi\lambda/p)} + e^{-in(\tau+2\pi\lambda/p)}), \\
&\ddot{\sigma}_\lambda^{(1)} + 2\Omega\dot{\rho}_\lambda^{(1)} = \frac{1}{a^2(\omega-\omega')^2}\left[\sum_\mu\left(\frac{\partial^2\Theta_\lambda}{\partial\theta_\lambda \partial r_\mu}\right)_0 a\rho_\mu^{(1)} + \sum_\mu\left(\frac{\partial^2\Theta_\lambda}{\partial\theta_\lambda\partial\theta_\mu}\right)_0 \sigma_\mu^{(1)}\right] \\
&\qquad\qquad + \frac{i}{2a^2a'(\omega-\omega')^2}\sum_n nB_n(e^{in(\tau+2\pi\lambda/p)} - e^{-in(\tau+2\pi\lambda/p)}).
\end{aligned}\right\} \quad (9)$$

For convenience the upper suffix may be dropped temporarily.

Equations (9) are $2p$ in number each involving every co-ordinate ρ_μ, σ_μ $(\mu = 1, 2, \ldots, p)$. They may be reduced to a single pair by the device introduced by Poincaré (1900) and used by Pendse (1935). For any integer s between 1 and p inclusive, we put

$$\left.\begin{aligned} \xi_s &= \frac{1}{p}\sum_{\lambda=1}^{p} \rho_\lambda\, e^{-2\pi i s\lambda/p}, \\ \eta_s &= \frac{1}{p}\sum_{\lambda=1}^{p} \sigma_\lambda\, e^{-2\pi i s\lambda/p}. \end{aligned}\right\} \tag{10}$$

From these it follows that

$$\left.\begin{aligned} \rho_\lambda &= \sum_{s=1}^{p} \xi_s\, e^{2\pi i s\lambda/p}, \\ \sigma_\lambda &= \sum_{s=1}^{p} \eta_s\, e^{2\pi i s\lambda/p}. \end{aligned}\right\} \tag{11}$$

Now multiply each λ-equation of (9) by $\dfrac{1}{p}e^{-2\pi i s\lambda/p}$ and sum the sets so formed separately. In the summations the following results will be required:

$$\sum_\lambda \frac{1}{p}\left[-2K\rho_\lambda + \sum_\mu \left(\frac{\partial^2 \Theta_\lambda}{\partial r_\lambda \partial r_\mu}\right)_0 \rho_\mu\right] e^{-2\pi i s\lambda/p}$$
$$= -\frac{m}{8a^3}\sum_{n=1}^{p-1}(\operatorname{cosec}^3 n\pi/p + \operatorname{cosec} n\pi/p)\,\xi_s$$
$$+\frac{m}{a^3}\sum_{n=1}^{p-1}\left(\frac{1+\sin^2 n\pi/p}{8\sin^3 n\pi/p} + 2\cos 2n\pi/p\right)\cos 2\pi s n/p\,\xi_s,$$

which we write as $\nu(N+P_s)\,\omega^2\xi_s$, ν being put for $\dfrac{m/M}{1+Km/M}$, so that from (2) $\dfrac{m}{a^3} = \nu\omega^2$;

$$\frac{1}{a}\sum_\lambda \frac{1}{p}\sum_\mu \left(\frac{\partial^2 \Theta_\lambda}{\partial r_\lambda \partial \theta_\mu}\right)_0 \sigma_\mu\, e^{-2\pi i s\lambda/p}$$
$$= i\frac{m}{a^3}\sum_{n=1}^{p-1}(\tfrac{1}{16}\operatorname{cosec}^3 n\pi/p + 1)\sin 2n\pi/p \sin 2\pi s n/p\,\eta_s$$
$$= i\nu Q_s\omega^2\eta_s;$$

$$\frac{1}{a}\sum_\lambda\sum_\mu \frac{1}{p}\left(\frac{\partial^2 \Theta_\lambda}{\partial \theta_\lambda \partial r_\mu}\right)_0 \rho_\mu\, e^{-2\pi i s\lambda/p} = \frac{m}{a^3}\left[\sum_{n=1}^{p-1}\left(\frac{1+\cos^2 n\pi/p}{8\sin^3 n\pi/p} + \cos 2n\pi/p\right)\right.$$
$$\left. - \sum_{n=1}^{p-1}\left(\frac{1+\cos^2 n\pi/p}{8\sin^3 n\pi/p} + \cos 2n\pi/p\right)\cos 2ns\pi/p\right]\xi_s$$
$$= \nu T_s\omega^2\xi_s;$$

$$\frac{1}{a^2}\sum_\lambda\sum_\mu \frac{1}{p}\left(\frac{\partial^2 \Theta_\lambda}{\partial \theta_\lambda \partial \theta_\mu}\right)_0 \sigma_\mu\, e^{-2\pi i s\lambda/p} = i\frac{m}{a^3}\sum_{n=1}^{p-1}(2 - \tfrac{1}{16}\operatorname{cosec}^3 n\pi/p)\sin 2n\pi/p \sin 2\pi s n/p\,\eta_s$$
$$= i\nu R_s\omega^2\eta_s;$$

$$\sum_{\lambda=1}^{p}\sum_n \frac{1}{p} B'_n e^{i(n\tau + 2\pi n\lambda/p - 2\pi s\lambda/p)} = \frac{1}{p}\sum_n B'_n e^{in\tau}\sum_{\lambda=1}^{p} e^{i\lambda(n-s)2\pi/p}$$
$$= 0, \text{ unless } (n-s) \text{ is zero or an integral multiple of } p.$$

Put $n = mp+s$, $m = 0, 1, 2, \ldots$.* Then

$$\sum_{\lambda=1}^{p}\sum_{n}\frac{1}{p}B'_n e^{i(n\tau+2\pi n\lambda/p-2\pi s\lambda/p)} = \sum_{m=0}^{\infty} B'_{mp+s}e^{i(mp+s)\tau}.$$

Similarly, $$\sum_{\lambda=1}^{p}\sum_{n}\frac{1}{p}B'_n e^{-i(n\tau+2\pi n\lambda/p+2\pi s\lambda/p)} = \sum_{m=1}^{\infty} B'_{mp-s}e^{-i(mp-s)\tau},$$

$$\sum_{\lambda=1}^{p}\sum_{n}\frac{1}{p}nB_n e^{i(n\tau+2\pi n\lambda/p-2\pi s\lambda/p)} = \sum_{m=0}^{\infty} B_{mp+s}(mp+s)\,e^{i(mp+s)\tau},$$

and $$\sum_{\lambda=1}^{p}\sum_{n}\frac{1}{p}nB_n e^{-i(n\tau+2\pi n\lambda/p+2\pi s\lambda/p)} = \sum_{m=1}^{\infty} B_{mp-s}(mp-s)\,e^{-i(mp-s)\tau}.$$

From these, equations (9) become

$$\left.\begin{aligned}
&\ddot{\xi}_s-2\Omega\dot{\eta}_s-(3\Omega^2+\nu\Omega^2N+\nu\Omega^2P_s)\,\xi_s-i\nu\Omega^2Q_s\eta_s\\
&\qquad=\frac{1}{2aa'^2(\omega-\omega')^2}\left[\sum_{m=0}^{\infty}B'_{mp+s}e^{i(mp+s)\tau}+\sum_{m=1}^{\infty}B'_{mp-s}e^{-i(mp-s)\tau}\right],\\
&\ddot{\eta}_s+2\Omega\dot{\xi}_s-\nu\Omega^2T_s\eta_s-i\nu\Omega^2R_s\xi_s\\
&\qquad=\frac{1}{2a^2a'(\omega-\omega')^2}\left[\sum_{m=0}^{\infty}B_{mp+s}(mp+s)\,e^{i(mp+s)\tau}\right.\\
&\qquad\qquad\left.-\sum_{m=1}^{\infty}B_{mp-s}(mp-s)\,e^{-i(mp-s)\tau}\right].
\end{aligned}\right\}\tag{12}$$

The particular integral of these equations is

$$\left.\begin{aligned}
\xi_s&=\sum_{m=0}^{\infty}X_{mp+s}e^{i(mp+s)\tau}+\sum_{m=1}^{\infty}X_{-mp+s}e^{-i(mp-s)\tau},\\
\eta_s&=\sum_{m=0}^{\infty}iY_{mp+s}e^{i(mp+s)\tau}+\sum_{m=1}^{\infty}iY_{-mp+s}e^{-i(mp-s)\tau},
\end{aligned}\right\}\tag{13}$$

where

$$X_k=\frac{1}{2a^2a'^2(\omega-\omega')^2}\{aB'_k(k^2+\nu\Omega T_k)+a'kB_k(2k\Omega+\nu\Omega^2Q_k)\}\div D^k_k(\Omega),$$

$$Y_k=\frac{1}{2a^2a'^2(\omega-\omega')^2}\{aB'_k(2k\Omega-\nu\Omega^2R_k)+a'B_k(k^2+3\Omega^2+\nu\Omega^2N+\nu\Omega^2P_s)\}\div D^k_k(\Omega),$$

$$D^s_k(\Omega)=(2k\Omega-\nu\Omega^2R_s)\,(2k\Omega+\nu\Omega^2Q_s)-(k^2+\nu\Omega T_s)\,(k^2+3\Omega^2+\nu\Omega^2N+\nu\Omega^2P_s).\tag{14}$$

* The use of m here as an integer will not be confused with its previous use as a mass.

In the particular case where $s = p$ we have in place of equations (12)

$$\left.\begin{aligned} \ddot{\xi}_p - 2\Omega\dot{\eta}_p - 3\Omega^2\xi_p &= \frac{B'_0}{aa'^2(\omega-\omega')^2}, \\ \ddot{\eta}_p + 2\Omega\dot{\xi}_p &= 0. \end{aligned}\right\}$$

These give the particular integral

$$\left.\begin{aligned} \xi_p &= -\frac{B'_0}{3\Omega^2 aa'^2(\omega-\omega')^2}, \\ \eta_p &= 0. \end{aligned}\right\} \tag{15}$$

It is to be noted that for m any integer including zero, and p large,

$$P_s = P_{mp\pm s}, \quad Q_s = \pm Q_{mp\pm s}, \quad R_s = \pm R_{mp\pm s}, \quad T_s = T_{mp\pm s};$$

also $$B_s = B_{-s}, \quad B'_s = B'_{-s}, \quad N + P_p = 0, \quad Q_p = R_p = T_p = 0.$$

Hence $$X_{-k} = X_k, \quad Y_{-k} = Y_k.$$

The solutions (13) and (15) are periodic of the required form, the new period in τ being 2π. But the solution fails when $D^s_{\pm mp+s} = 0$, as terms involving a factor τ appear. Hence for those values of Ω satisfying $D^s_{\pm mp+s}(\Omega) = 0$ no periodic solutions of the type required occur. But the periodic solution found in (13) and (15) exists at all other values of Ω.

To determine the complementary function we omit the right-hand numbers of equations (12). The equations, so reduced, are exactly those used by Maxwell to determine the stability of a ring of particles with no satellite present. On putting $\xi, \eta \propto \epsilon^{i\gamma\tau}$ we find the characteristic equation

$$D^s_\gamma(\Omega) = 0,$$

or $$\left(\frac{\gamma}{\Omega}\right)^4 + \left(\frac{\gamma}{\Omega}\right)^2 \{\nu(N+P_s+T_s) - 1\} + 2\frac{\gamma}{\Omega}\nu(R_s - Q_s)$$
$$+\nu T_s(3+\nu N+\nu P_s) + \nu^2 Q_s R_s = 0. \tag{16}$$

Maxwell showed that all four roots of this equation were real if ν were small enough. We shall assume this condition satisfied, so that the ring of particles is stable, apart from the influence of the satellite. Since we require only solutions with a period 2π, γ must be an integer, say k. With k and s fixed, equation (16) gives four real values of Ω. At the positions defined by these values of Ω, there will be at least one periodic solution in the complementary function of the form $A\epsilon^{ik\tau}$, with A an arbitrary constant. At other values of Ω there will be no periodic solution arising from the complementary function and having this form.

Since values of Ω satisfying $D^s_{\pm mp+s}(\Omega) = 0$ are already excluded, k cannot have any of the values s, $\pm p + s$, $\pm 2p + s$, ...

In compiling the complete solution of equations (9) we have

$$\rho_\lambda = \sum_{s=1}^{p} e^{2\pi i\lambda s/p}\,\xi_s$$

$$= \sum_{s=1}^{p}\left[\sum_{m=0}^{\infty} X_{mp+s}\,e^{i(mp\tau+s\tau+2\pi\lambda s/p)} + \sum_{m=1}^{\infty} X_{-mp+s}\,e^{i(-mp\tau+s\tau+2\pi\lambda s/p)}\right]$$

$$= X_0 + \sum_{s=1}^{\infty} X_s\,e^{is(\tau+2\pi\lambda/p)} + \sum_{s=1}^{\infty} X_{-s}\,e^{-is(\tau+2\pi\lambda/p)}$$

$$= X_0 + 2\sum_{s=1}^{\infty} X_s \cos s(\tau+2\pi\lambda/p).$$

Hence the final form of the solution is

$$\left.\begin{aligned}\rho_\lambda^{(1)} &= X_0^{(1)} + 2\sum_{s=1}^{\infty} X_s^{(1)}\cos s(\tau+2\pi\lambda/p) + A_s^{(1)}\,e^{i(k\tau+2\pi\lambda s/p)},\\ \sigma_\lambda^{(1)} &= \qquad -2\sum_{s=1}^{\infty} Y_s^{(1)}\sin s(\tau+2\pi\lambda/p) + B_s^{(1)}\,e^{i(k\tau+2\pi\lambda s/p)},\\ &B_s^{(1)}(\nu\Omega^2T_s+k^2) = iA_s^{(1)}(2\Omega k-\nu\Omega^2R_s).\end{aligned}\right\} \tag{17}$$

This solution is subject to two restrictions: (*a*) Ω must not be a solution of $D^s_{\pm mp+s}(\Omega) = 0$ for any integer s from 1 to p and any integer m including zero; and (*b*) A_s is zero unless Ω satisfies $D^s_k(\Omega) = 0$ for selected s.

Only one term arising from the complementary function is included in (17). In special cases there may be others. But as the associated arbitrary constants have to be determined at a later stage, and as it appears that for a periodic solution these constants must be zero, it is sufficient to include one term as an example.

The convergence of (17) must be examined. The series

$$\Sigma B_n \cos n\theta, \quad \Sigma B_n \sin n\theta, \quad \Sigma B_n' \cos n\theta, \quad \Sigma B_n' \sin n\theta,$$

are well known to be convergent for $|\alpha|<1$, $0\leqslant\theta\leqslant 2\pi$. For a fixed value of Ω, the limiting form of $X_s^{(1)}$ is

$$\frac{1}{2a^2a'^2(\omega-\omega')^2}\{aB_s'+2a'\Omega B_s\}\div s^2$$

with a similar form for $Y_s^{(1)}$.

Hence the series for ρ_λ, σ_λ are convergent under the same conditions.

THE HIGHER TERMS OF THE SOLUTION

It is necessary to examine the general character of the terms of the solution associated with higher powers of ϵ. For convenience, write

$$\left.\begin{aligned}L(\rho_\lambda^{(n)}, \sigma_\lambda^{(n)}) \equiv{} & \ddot{\rho}_\lambda^{(n)} - 2\Omega\dot{\sigma}_\lambda^{(n)} - 3\Omega^2\rho_\lambda^{(n)} + \frac{2K}{(\omega-\omega')^2}\rho_\lambda^{(n)} \\ & -\frac{1}{a(\omega-\omega')^2}\left[\sum_\mu \left(\frac{\partial^2\Theta_\lambda}{\partial r_\lambda \partial r_\mu}\right)_0 a\rho_\mu^{(n)} + \sum_\mu \left(\frac{\partial^2\Theta_\lambda}{\partial r_\lambda \partial \theta_\mu}\right)_0 \sigma_\mu^{(n)}\right], \\ M(\rho_\lambda^{(n)}, \sigma_\lambda^{(n)}) \equiv{} & \ddot{\sigma}_\lambda^{(n)} + 2\Omega\dot{\rho}_\lambda^{(n)} \\ & -\frac{1}{a^2(\omega-\omega')^2}\left[\sum_\mu \left(\frac{\partial^2\Theta_\lambda}{\partial \theta_\lambda \partial r_\mu}\right)_0 a\rho_\mu^{(n)} + \sum_\mu \left(\frac{\partial^2\Theta_\lambda}{\partial \theta_\lambda \partial \theta_\mu}\right)_0 \sigma_\mu^{(n)}\right].\end{aligned}\right\} \quad (18)$$

Then, taking the terms factored by ϵ^2 from (3)...(8), we have

$$\left.\begin{aligned}L(\rho_\lambda^{(2)}, \sigma_\lambda^{(2)}) ={} & \frac{1}{2aa'^2(\omega-\omega')^2}\sum_n B_n'(e^{in(\tau+2\pi\lambda/p)} + e^{-in(\tau+2\pi\lambda/p)}) \\ & + 2\Omega\dot{\sigma}_\lambda^{(1)}\rho_\lambda^{(1)} + (\dot{\sigma}_\lambda^{(1)})^2 - 3\Omega^2(\rho_\lambda^{(1)})^2 + 3K(\omega-\omega')^2(\rho_\lambda^{(1)})^2 \\ & + \frac{1}{a(\omega-\omega')^2}\left\{\sum \left(\frac{\partial^3\Theta_\lambda}{\partial r_\lambda \partial r_\mu \partial r_\kappa}\right)_0 a^2\rho_\mu^{(1)}\rho_\kappa^{(1)} + \sum \left(\frac{\partial^3\Theta_\lambda}{\partial r_\lambda \partial r_\mu \partial \theta_\kappa}\right)_0 a\rho_\mu^{(1)}\sigma_\kappa^{(1)}\right\} \\ & + \ldots, \\ M(\rho_\lambda^{(2)}, \sigma_\lambda^{(2)}) ={} & \frac{1}{2a^2a'(\omega-\omega')^2}\sum_n nB_n(e^{in(\tau+2\pi\lambda/p)} - e^{-in(\tau+2\pi\lambda/p)}) \\ & - 2\Omega\rho_\lambda^{(1)}\dot{\rho}_\lambda^{(1)} - 2\dot{\rho}_\lambda^{(1)}\dot{\sigma}_\lambda^{(1)} - 2\ddot{\sigma}_\lambda^{(1)}\rho_\lambda^{(1)} \\ & + \frac{1}{a^2(\omega-\omega')^2}\left\{\sum \left(\frac{\partial^3\Theta_\lambda}{\partial \theta_\lambda \partial r_\mu \partial r_\kappa}\right)_0 a^2\rho_\mu^{(1)}\rho_\kappa^{(1)} + \sum \left(\frac{\partial^3\Theta_\lambda}{\partial \theta_\lambda \partial r_\mu \partial \theta_\kappa}\right)_0 a\rho_\mu^{(1)}\sigma_\kappa^{(1)}\right\} \\ & + \ldots,\end{aligned}\right\} \quad (19)$$

In these equations, only typical terms have been written down in the right-hand members.

The left members of equations (19) and the first terms of the right members are exactly the same as appear in equations (9). Hence the complementary function and part of the particular integral will be the same for $\rho_\lambda^{(2)}$, $\sigma_\lambda^{(2)}$, as for $\rho_\lambda^{(1)}$, $\sigma_\lambda^{(1)}$. New terms will appear in the particular integral arising from the remaining terms of the right-hand members. Consider the part of the particular integral arising from the term $2\Omega\dot{\sigma}_\lambda^{(1)}\rho_\lambda^{(1)}$. On substituting from (17) it is found that terms of the form $\cos(m-n)(\tau+2\pi\lambda/p)$ and $\cos(m+n)(\tau+2\pi\lambda/p)$ appear. These are of the same type as those already in the particular integral. In addition, there will be, for four special values of Ω, a term $ikX_0^{(1)}B_s^{(1)}e^{i(k\tau+2\pi\lambda s/p)}$. Since for these values of Ω, $e^{ik\tau}$ is part of the complementary function, this term will give rise to non-periodic terms in the particular integral unless

$B_s^{(1)} = 0$. Hence the arbitrary constant arising in (17) must be zero; and a similar argument will apply should there be more terms with arbitrary constants arising from the complementary function of (9).

Further, it is clear that the only types of terms appearing in the right-hand members of (19) are circular functions of multiples of $(\tau + 2\pi\lambda/p)$. A single example will suffice. Consider the term

$$\sum_\mu \sum_\kappa \left(\frac{\partial^3 \Theta_\lambda}{\partial r_\lambda \partial r_\mu \partial r_\kappa}\right)_0 \rho_\mu^{(1)} \rho_\kappa^{(1)}.$$

On differentiating and reducing, it exhibits three groups of terms having the forms

$$\sum_\mu{}' F_1(\lambda-\mu)\, X_m X_n\, e^{i(m+n)\tau + i(m+n)\, 2\pi\mu/p},$$

$$\sum_\mu{}' F_2(\lambda-\mu)\, X_m X_n\, e^{i(m+n)\tau + i(m\lambda+n\mu)\, 2\pi/p},$$

$$\sum_\mu{}' F_3(\lambda-\mu)\, X_m X_n\, e^{i(m+n)\tau + i(m+n)\, 2\pi\lambda/p},$$

where each $F(\lambda-\mu)$ is a definite function of $(\lambda-\mu)$ and such that

$$F(\lambda-\mu) = F(\pm p + \lambda - \mu),$$

and X is independent of μ. Then

$$\sum_\mu{}' F_1(\lambda-\mu)\, X_m X_n\, e^{i(m+n)\tau + i(m+n)\, 2\pi\mu/p}$$
$$= \sum_\mu{}' F_1(\lambda-\mu)\, e^{i(m+n)(\mu-\lambda)\, 2\pi/p} X_m X_n\, e^{i(m+n)(\tau + 2\pi\lambda/p)}.$$

Since the coefficient $\sum_\mu{}' F_1(\lambda-\mu)\, e^{i(m+n)(\mu-\lambda)\,2\pi/p}$ is independent of λ and μ, the function has the form, a constant $\times e^{i(m+n)(\tau+2\pi\lambda/p)}$.

Similarly, $$\sum_\mu{}' F_2(\lambda-\mu)\, X_m X_n\, e^{i(m+n)\tau + i(m\lambda+n\mu)\, 2\pi/p}$$
$$= \sum_\mu{}' F_2(\lambda-\mu)\, e^{in(\mu-\lambda)\, 2\pi/p} X_m X_n\, e^{i(m+n)(\tau+2\pi\lambda/p)},$$

which has the same characteristic form.

The last expression is already in the appropriate form. Since then the only types of terms appearing in the right-hand members of (19) are of the form $e^{\pm in(\tau+2\pi\lambda/p)}$, it follows that the solution of (19) will be similar to that of (9), viz.

$$\left.\begin{aligned} \rho_\lambda^{(2)} &= X_0^{(2)} + 2\sum_{s=1}^{\infty} X_s^{(2)} \cos s(\tau + 2\pi\lambda/p) + A_s^{(2)}\, e^{i(k\tau + 2\pi\lambda s/p)}, \\ \sigma_\lambda^{(2)} &= \qquad -2\sum_{s=1}^{\infty} Y_s^{(2)} \sin s(\tau + 2\pi\lambda/p) + B_s^{(2)}\, e^{i(k\tau + 2\pi\lambda s/p)}, \\ B_s^{(2)}(\nu\Omega T_s + k^2) &= iA_s^{(2)}(2\Omega k - \nu\Omega^2 R_s). \end{aligned}\right\} \qquad (20)$$

The values of $X_s^{(2)}$, $Y_s^{(2)}$ are more complicated than those of $X_s^{(1)}$, $Y_s^{(1)}$. But they will have the denominator $D_s^s(\Omega)$ and may have also another denominator $D_n^n(\Omega)$, where n is any integer, including s.

On proceeding to the terms in ϵ^3, it will be found as before that for a periodic solution the arbitrary constant $A_s^{(2)} = 0$, and that again only terms in the form of circular functions of multiples of $\tau + 2\pi\lambda/p$ appear in the right-hand members of equations associated with ρ_λ, σ_λ. The solution at this stage will also have the form of (17) or (20) with new coefficients $X_s^{(3)}$, $Y_s^{(3)}$ appearing.

The complete solution of equations (1) having the period 2π is therefore

$$\left.\begin{aligned} r_\lambda/a &= 1 + \sum_1^\infty \epsilon^n X_0^{(n)} + 2\sum_{n=1}^\infty \epsilon^n \sum_{s=1}^\infty X_s^{(n)} \cos s(\tau + 2\pi\lambda/p), \\ \theta_\lambda &= \tau + 2\pi\lambda/p \quad - 2\sum_{n=1}^\infty \epsilon^n \sum_{s=1}^\infty Y_s^{(n)} \sin s(\tau + 2\pi\lambda/p). \end{aligned}\right\} \tag{21}$$

This solution is valid for all values of Ω except those satisfying the equation $D^s_{\pm mp+s}(\Omega) = 0$, m, s being integers. From the theorems previously mentioned, the series are convergent for all time provided ϵ is sufficiently small, except at the points defined by $D^s_{\pm mp+s}(\Omega) = 0$.

The positions at which the periodic solutions do not exist are analogous to those appearing in similar problems. On putting ν indefinitely small, the equation $D_s^s(\Omega) = 0$ reduces to

$$s^2(\Omega^2 - s^2) = 0,$$

giving

$$\Omega = \pm s,$$

or

$$\frac{\omega'}{\omega} = \frac{s+1}{s},\ \frac{s-1}{s}.$$

This is in agreement with known results in the satellite problem (Moulton 1920).

The roots of the characteristic equation

To determine the positions at which no periodic orbits exist, and for other purposes, we require some knowledge of the roots of the algebraic equation (16) which is

$$\left(\frac{\gamma}{\Omega}\right)^4 + \left(\frac{\gamma}{\Omega}\right)^2 \{\nu(N + P_s + T_s) - 1\} + 2\left(\frac{\gamma}{\Omega}\right)\nu(R_s - Q_s) + \nu T_s(3 + \nu N + \nu P_s) + \nu^2 Q_s R_s = 0. \tag{22}$$

We only need to consider the case applicable to the problem, that is, the case when the ring of particles without a satellite is stable. As already mentioned, this case occurs when ν is sufficiently small. We shall suppose this condition fulfilled. Putting $\nu = 0$ in (22) we have $\gamma/\Omega = 0,\ 0,\ \pm 1$, as the four roots. From the general theorem on equations it follows that the roots can be expressed as power series in $\nu^{\frac{1}{2}}$ for the first pair and ν for the last pair. On working these out we have, with sufficient accuracy for our purpose,

$$\gamma_1/\Omega = 1-\tfrac{1}{2}\nu(N+P_s+T_s+2R_s-2Q_s)+\dots, \tag{23}$$

$$\gamma_2/\Omega = -1+\tfrac{1}{2}\nu(N+P_s+T_s-2R_s+2Q_s)+\dots, \tag{24}$$

$$\gamma_3/\Omega = \sqrt{(3\nu T_s)}+\nu(R_s-Q_s)+\dots, \tag{25}$$

$$\gamma_4/\Omega = -\sqrt{(3\nu T_s)}+\nu(R_s-Q_s)+\dots. \tag{26}$$

The positions at which $D^s_s(\Omega) = 0$ are therefore

$$\Omega = s\{1-\tfrac{1}{2}\nu(N+P_s+4T_s+2R_s-2Q_s)+\dots\}^{-1}, \tag{27}$$

$$\Omega = s\{-1+\tfrac{1}{2}\nu(N+P_s+4T_s-2R_s+2Q_s)+\dots\}^{-1}, \tag{28}$$

$$\Omega = s\{\sqrt{(3\nu T_s)}+\dots\}^{-1}, \tag{29}$$

$$\Omega = -s\{\sqrt{(3\nu T_s)}+\dots\}^{-1}. \tag{30}$$

$$(s = 1, 2, 3, \dots, p).$$

Values of Ω which are negative will be rejected, as they refer to positions outside the orbit of the perturbing satellite and are therefore of no physical interest. It is obvious that under the assumed conditions, equation (22) cannot have a pair of equal roots, but two roots may differ by an integer. Consider the more general case of the latter supposition. Suppose we have

$$D^s_\gamma(\Omega) = 0,$$

and

$$D^n_{\gamma+q}(\Omega) = 0, \quad (n>s)$$

for the same value (or values) of Ω, q being an integer. Then from the preceding results we have, keeping only those which produce positive values of Ω,

$$\left.\begin{array}{ll}
\text{(i)} & \gamma/\Omega = -1+\nu V_n+\dots, \\
& \Omega = q\{2-\nu(U_s+V_n)+\dots\}^{-1}; \\
\text{(ii)} & \gamma/\Omega = -1+\nu V_s+\dots, \\
& \Omega = q\{2-\nu(V_s+U_n)+\dots\}^{-1}; \\
\text{(iii)} & \gamma/\Omega = \sqrt{(3\nu T_n)}+\dots, \\
& \Omega = q\{\sqrt{(3\nu T_n)}-\sqrt{(3\nu T_s)}\}^{-1}; \\
\text{(iv)} & \gamma/\Omega = -\sqrt{(3\nu T_n)}+\dots, \\
& \Omega = q\{\sqrt{(3\nu T_n)}+\sqrt{(3\nu T_s)}\}^{-1}; \\
\text{(v)} & \gamma/\Omega = \sqrt{(3\nu T_n)}+\dots, \\
& \Omega = q\{1-\sqrt{(3\nu T_n)}+\dots\}^{-1}; \\
\text{(vi)} & \gamma/\Omega = -\sqrt{(3\nu T_n)}+\dots, \\
& \Omega = q\{1+\sqrt{(3\nu T_n)}+\dots\}^{-1}.
\end{array}\right\} \tag{31}$$

For convenience we have written

$$U_s \equiv \tfrac{1}{2}(N + P_s + 4T_s + 2R_s - 2Q_s),$$

and

$$V_s \equiv \tfrac{1}{2}(N + P_s + 4T_s - 2R_s + 2Q_s).$$

Each of these solutions will be required in the sequel.

The variational equations

We next proceed to study the stability of the periodic solutions just determined, using the term stability in the sense defined by Poincaré (1892).

In the fundamental equations (1) substitute

$$r_\lambda = r_\lambda^{(0)} + ax_\lambda, \quad \theta_\lambda = \theta_\lambda^{(0)} + y_\lambda,$$

where $r_\lambda^{(0)}$, $\theta_\lambda^{(0)}$ are given in the periodic solution (21) and first powers of x_λ, y_λ only are retained.

The equations then reduce to

$$\left.\begin{aligned} & a\frac{d^2x_\lambda}{dt^2} - ax_\lambda\left(\frac{d\theta_\lambda^{(0)}}{dt} + \omega'\right)^2 - 2r_\lambda^{(0)}\frac{dy_\lambda}{dt}\left(\frac{d\theta_\lambda^{(0)}}{dt} + \omega'\right) - \frac{2Max_\lambda}{(r_\lambda^{(0)})^2} \\ & \qquad = \sum_\mu ax_\mu\left(\frac{\partial^2\Theta_\lambda}{\partial r_\lambda \partial r_\mu}\right)_0 + \sum_\mu y_\mu\left(\frac{\partial^2\Theta_\lambda}{\partial r_\lambda \partial \theta_\mu}\right)_0 + \epsilon ax_\lambda\left(\frac{\partial^2\Phi_\lambda}{\partial r_\lambda^2}\right)_0 + \epsilon y_\lambda\left(\frac{\partial^2\Phi_\lambda}{\partial r_\lambda \partial \theta_\lambda}\right)_0, \\ & (r_\lambda^{(0)})^2\frac{d^2y_\lambda}{dt^2} + 2ar_\lambda^{(0)}\frac{d^2\theta_\lambda^{(0)}}{dt^2}x_\lambda + 2r_\lambda^{(0)}\frac{dr_\lambda^{(0)}}{dt}\frac{dy_\lambda}{dt} \\ & \qquad + 2ax_\lambda\frac{dr_\lambda^{(0)}}{dt}\frac{d\theta_\lambda^{(0)}}{dt} + 2ax_\lambda\omega'\frac{dr_\lambda^{(0)}}{dt} + 2ar_\lambda^{(0)}\left(\frac{d\theta_\lambda^{(0)}}{dt} + \omega'\right)\frac{dx_\lambda}{dt} \\ & \qquad = \sum_\mu ax_\mu\left(\frac{\partial^2\Theta_\lambda}{\partial \theta_\lambda \partial r_\mu}\right)_0 + \sum_\mu y_\mu\left(\frac{\partial^2\Theta_\lambda}{\partial \theta_\lambda \partial \theta_\mu}\right)_0 + \epsilon ax_\lambda\left(\frac{\partial^2\Phi_\lambda}{\partial \theta_\lambda \partial r_\lambda}\right)_0 + \epsilon y_\lambda\left(\frac{\partial^2\Phi_\lambda}{\partial \theta_\lambda^2}\right)_0. \end{aligned}\right\} \quad (32)$$

The zero suffix to the derivatives implies that, after differentiating, the substitution $r_\lambda = r_\lambda^{(0)}$, $\theta_\lambda = \theta_\lambda^{(0)}$ is made for each λ.

Divide equations (32) by $(\omega - \omega')^2$ and change the independent variable to τ, $= t(\omega - \omega')$, as before. Also substitute

$$r_\lambda^{(0)} = a(1 + \epsilon\rho_\lambda^{(1)} + \epsilon^2\rho_\lambda^{(2)} + \ldots),$$

$$\theta_\lambda^{(0)} = \tau + 2\pi\lambda/p + \epsilon\sigma_\lambda^{(1)} + \epsilon^2\sigma_\lambda^{(2)} + \ldots.$$

Equations (32) then become

$$\left.\begin{aligned}
&\ddot{x}_\lambda - 2\Omega\dot{y}_\lambda - x_\lambda\Omega^2 - \frac{2Mx_\lambda}{a^3(\omega-\omega')^2} - \sum_\mu \frac{x_\mu}{(\omega-\omega')^2}\left(\frac{\partial^2\Theta_\lambda}{\partial r_\lambda \partial r_\mu}\right)_{00} - \sum_\mu \frac{y_\mu}{(\omega-\omega')^2}\left(\frac{\partial^2\Theta_\lambda}{\partial r_\lambda \partial \theta_\mu}\right)_{00} \\
&\quad = \epsilon\Big[2\Omega x_\lambda \dot{\sigma}_\lambda^{(1)} + 2\dot{y}_\lambda \dot{\sigma}_\lambda^{(1)} + 2\Omega\rho_\lambda^{(1)}\dot{y}_\lambda - \frac{6Mx_\lambda\rho_\lambda^{(1)}}{a^3(\omega-\omega')^2} \\
&\qquad + \sum \frac{x_\mu}{(\omega-\omega')^2}\left\{\left(\frac{\partial^3\Theta_\lambda}{\partial r_\lambda \partial r_\mu \partial r_\kappa}\right)_{00} a\rho_\kappa^{(1)} + \left(\frac{\partial^3\Theta_\lambda}{\partial r_\lambda \partial r_\mu \partial \theta_\kappa}\right)_{00} \sigma_\kappa^{(1)}\right\} \\
&\qquad + \sum \frac{y_\mu}{a(\omega-\omega')^2}\left\{\left(\frac{\partial^3\Theta_\lambda}{\partial r_\lambda \partial \theta_\mu \partial r_\kappa}\right)_{00} a\rho_\kappa^{(1)} + \left(\frac{\partial^3\Theta_\lambda}{\partial r_\lambda \partial \theta_\mu \partial \theta_\kappa}\right)_{00} \sigma_\kappa^{(1)}\right\} \\
&\qquad + \frac{x_\lambda}{(\omega-\omega')^2}\left(\frac{\partial^2\Phi_\lambda}{\partial r_\lambda^2}\right)_{00} + \frac{y_\lambda}{a(\omega-\omega')^2}\left(\frac{\partial^2\Phi_\lambda}{\partial r_\lambda \partial \theta_\lambda}\right)_{00}\Big] \\
&\qquad + \epsilon^2[\ldots] + \ldots, \\
\text{and}\quad &\ddot{y}_\lambda + 2\Omega\dot{x}_\lambda - \sum_\mu \frac{x_\mu}{a(\omega-\omega')^2}\left(\frac{\partial^2\Theta_\lambda}{\partial \theta_\lambda \partial r_\mu}\right)_{00} - \sum_\mu \frac{y_\mu}{a^2(\omega-\omega')^2}\left(\frac{\partial^2\Theta_\lambda}{\partial \theta_\lambda \partial \theta_\mu}\right)_{00} \\
&\quad = \epsilon\Big[-2\rho_\lambda^{(1)}\ddot{y}_\lambda - 2x_\lambda\ddot{\sigma}_\lambda^{(1)} - 2\dot{y}_\lambda\dot{\rho}_\lambda^{(1)} - 2\Omega x_\lambda\dot{\rho}_\lambda^{(1)} - 2\Omega\dot{x}_\lambda\rho_\lambda^{(1)} - 2\dot{x}_\lambda\dot{\sigma}_\lambda^{(1)} \\
&\qquad + \sum \frac{x_\mu}{a(\omega-\omega')^2}\left\{\left(\frac{\partial^3\Theta_\lambda}{\partial \theta_\lambda \partial r_\mu \partial r_\kappa}\right)_{00} a\rho_\kappa^{(1)} + \left(\frac{\partial^3\Theta_\lambda}{\partial \theta_\lambda \partial r_\mu \partial \theta_\kappa}\right)_{00} \sigma_\kappa^{(1)}\right\} \\
&\qquad + \sum \frac{y_\mu}{a^2(\omega-\omega')^2}\left\{\left(\frac{\partial^3\Theta_\lambda}{\partial \theta_\lambda \partial \theta_\mu \partial r_\kappa}\right)_{00} a\rho_\kappa^{(1)} + \left(\frac{\partial^3\Theta_\lambda}{\partial \theta_\lambda \partial \theta_\mu \partial \theta_\kappa}\right)_{00} \sigma_\kappa^{(1)}\right\} \\
&\qquad + \frac{x_\lambda}{a(\omega-\omega')^2}\left(\frac{\partial^2\Phi_\lambda}{\partial \theta_\lambda \partial r_\lambda}\right)_{00} + \frac{y_\lambda}{a^2(\omega-\omega')^2}\left(\frac{\partial^2\Phi_\lambda}{\partial \theta_\lambda^2}\right)_{00}\Big] \\
&\qquad + \epsilon^2[\ldots] + \ldots.
\end{aligned}\right\} \quad (33)$$

The values of the derivatives with the double zero suffix are obtained by putting $r_\lambda = a$, $\theta_\lambda = \tau + 2\pi\lambda/p$ after differentiating.

The right-hand members of (33) are known to be convergent series in ϵ, for values of ϵ sufficiently small, and the coefficients $\rho_\kappa^{(n)}$, $\sigma_\kappa^{(n)}$ are also known to be convergent series for $|\alpha|<1$, $0\leqslant\theta\leqslant 2\pi$.

The equations are a set of $2p$ linear equations of the second order with periodic coefficients, of period 2π. By Flocquet's theorem, the solution is known to be of the form

$$\left.\begin{aligned}
x_\lambda &= e^{c\tau}u_\lambda, \\
y_\lambda &= e^{c\tau}v_\lambda, \quad (\lambda = 1, 2, 3, \ldots, p)
\end{aligned}\right\} \quad (34)$$

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where u_λ, v_λ are periodic functions of period 2π, and c is a constant determined so that this condition is fulfilled.

Further, it is known that, by Poincaré's theorem (1892), we may express c, u_λ, v_λ as convergent series in powers of ϵ. That is,

$$\left.\begin{aligned} c &= ic_0+\epsilon c_1+\epsilon^2 c_2+\dots, \\ u_\lambda &= u_\lambda^{(0)}+\epsilon u_\lambda^{(1)}+\epsilon^2 u_\lambda^{(2)}+\dots, \\ v_\lambda &= v_\lambda^{(0)}+\epsilon v_\lambda^{(1)}+\epsilon^2 v_\lambda^{(2)}+\dots. \end{aligned}\right\} \tag{35}$$

The solution is determined by substituting (34) and (35) in (33) and equating to zero the various powers of ϵ.

Consider first the terms independent of ϵ. They are

$$\left.\begin{aligned} L(u_\lambda^{(0)}, v_\lambda^{(0)}, c_0) &\equiv \ddot{u}_\lambda^{(0)}+2ic_0\dot{u}_\lambda^{(0)}-c_0^2 u_\lambda^{(0)}-2\Omega(\dot{v}_\lambda^{(0)}+ic_0 v_\lambda^{(0)})-\Omega^2 u_\lambda^{(0)} \\ &\quad -\frac{2Mu_\lambda^{(0)}}{a^3(\omega-\omega')^2}-\sum_\mu \frac{u_\mu^{(0)}}{(\omega-\omega')^2}\left(\frac{\partial^2\Theta_\lambda}{\partial r_\lambda\,\partial r_\mu}\right)_{00}-\sum_\mu \frac{v_\mu^{(0)}}{a(\omega-\omega')^2}\left(\frac{\partial^2\Theta_\lambda}{\partial r_\lambda\,\partial \theta_\mu}\right)_{00}=0, \\ M(u_\lambda^{(0)}, v_\lambda^{(0)}, c_0) &\equiv \ddot{v}_\lambda^{(0)}+2ic_0\dot{v}_\lambda^{(0)}-c_0^2 v_\lambda^{(0)}+2\Omega(\dot{u}_\lambda^{(0)}+ic_0 v_\lambda^{(0)}) \\ &\quad -\sum_\mu \frac{u_\mu^{(0)}}{a(\omega-\omega')^2}\left(\frac{\partial^2\Theta_\lambda}{\partial \theta_\lambda\,\partial r_\mu}\right)_{00}-\sum_\mu \frac{v_\mu^{(0)}}{a^2(\omega-\omega')^2}\left(\frac{\partial^2\Theta_\lambda}{\partial \theta_\lambda\,\partial \theta_\mu}\right)_{00}=0. \end{aligned}\right\} \tag{36}$$

Multiply each equation by $\dfrac{1}{p}e^{-2\pi is\lambda/p}$ and sum each set with regard to λ, choosing s any integer from 1 to p.

If
$$\sum_{\lambda=1}^{p-1}(u_\lambda^{(0)}, v_\lambda^{(0)})\,e^{-2\pi is\lambda/p}=k_s^{(0)}, l_s^{(0)},$$
the equations become

$$\left.\begin{aligned} \overline{L}(k_s^{(0)}, l_s^{(0)}, c_0) &\equiv \ddot{k}_s^{(0)}+2ic_0\dot{k}_s^{(0)}-c_0^2 k_s^{(0)}-2\Omega(\dot{l}_s^{(0)}+ic_0 l_s^{(0)}) \\ &\quad -\Omega^2(3+\nu N+\nu P_s)\,k_s^{(0)}-i\nu\Omega^2 Q_s l_s^{(0)}=0, \\ \overline{M}(k_s^{(0)}, l_s^{(0)}, c_0) &\equiv \ddot{l}_s^{(0)}+2ic_0\dot{l}_s^{(0)}-c_0^2 l_s^{(0)}+2\Omega(\dot{k}_s^{(0)}+ic_0 k_s^{(0)}) \\ &\quad -i\nu\Omega^2 R_s k_s^{(0)}-\nu\Omega^2 T_s l_s^{(0)}=0, \end{aligned}\right\} \tag{37}$$

the values of N, P_s, Q_s, R_s, T_s being those already used. The component parameter c_0 has to be determined so that $k_s^{(0)}$, $l_s^{(0)}$ have period 2π.

In general, we may take the constant solution

$$\left.\begin{aligned} k_s^{(0)} &= \bar{k}_s^{(0)}, \\ l_s^{(0)} &= \bar{l}_s^{(0)}, \\ \bar{l}_s^{(0)} &= \frac{i\bar{k}_s^{(0)}(2\Omega c_0-\nu\Omega^2 R_s)}{c_0^2+\nu\Omega^2 T_s}. \end{aligned}\right\} \tag{38}$$

The equation to determine c_0 is then

$$D_c^s(\Omega) \equiv -\{c_0^2+\Omega^2(3+\nu N+\nu P_s)\}\{c_0^2+\nu\Omega^2 T_s\}+(2\Omega c_0+\nu\Omega^2 Q_s)(2\Omega c_0-\nu\Omega^2 R_s)=0. \quad (39)$$

This is simply equation (16). Under the conditions there mentioned it has four real roots, which have been detailed in (23), (24), (25), (26). The general procedure is to take each of these values of c_0 in turn and carry the solution to the next stage by considering the terms in ϵ. But three special cases need considering:

(i) The roots of (39) are all distinct whatever the value of s. The solution of (36) associated with one of the values of c_0 derived from (39) is then

$$\left.\begin{aligned} u_\lambda^{(0)} &= \bar{k}_s^{(0)}\, e^{2\pi i s\lambda/p}, \\ v_\lambda^{(0)} &= \bar{l}_s^{(0)}\, e^{2\pi i s\lambda/p}, \end{aligned}\right\} \quad (40)$$

the relation between $\bar{l}_s^{(0)}$ and $\bar{k}_s^{(0)}$ being (38). Only one arbitrary constant appears in this solution. As there are $4p$ distinct values of c_0, in the aggregate there will be $4p$ arbitrary constants, as is required.

(ii) The roots of (39) for a given s differ by an integer q. We have then for the same s and Ω,

$$D_{c_0}^s(\Omega)=0 \quad \text{and} \quad D_{c_0+q}^s(\Omega)=0.$$

The solution of (37) is then

$$\left.\begin{aligned} k_s^{(0)} &= \bar{k}_s^{(0)}+\bar{\bar{k}}_s^{(0)}\, e^{iq\tau}, \\ l_s^{(0)} &= \bar{l}_s^{(0)}+\bar{\bar{l}}_s^{(0)}\, e^{iq\tau}, \\ \text{with} \quad \bar{\bar{l}}_s^{(0)} &= \frac{i\bar{\bar{k}}_s^{(0)}\{2\Omega(c_0+q)-\nu\Omega^2 R_s\}}{(c_0+q)^2+\nu\Omega^2 T_s}. \end{aligned}\right\} \quad (41)$$

That is

$$\left.\begin{aligned} u_\lambda^{(0)} &= (\bar{k}_s^{(0)}+\bar{\bar{k}}_s^{(0)}\, e^{iq\tau})\, e^{2\pi i s\lambda/p}, \\ v_\lambda^{(0)} &= (\bar{l}_s^{(0)}+\bar{\bar{l}}_s^{(0)} e^{iq\tau})\, e^{2\pi i s\lambda/p}. \end{aligned}\right\} \quad (42)$$

(iii) The roots of the characteristic equation differ by an integer q for different values of s. That is

$$D_{c_0}^s(\Omega)=0 \quad \text{and} \quad D_{c_0+q}^n(\Omega)=0,$$

for the same value of Ω and $s \neq n$.

The resulting solution is

$$\left.\begin{aligned} u_\lambda^{(0)} &= \bar{k}_s^{(0)}\, e^{2\pi i s\lambda/p}+\bar{\bar{k}}_n^{(0)}\, e^{i(q\tau+2\pi n\lambda/p)}, \\ v_\lambda^{(0)} &= \bar{l}_s^{(0)}\, e^{2\pi i s\lambda/p}+\bar{\bar{l}}_n^{(0)}\, e^{i(q\tau+2\pi n\lambda/p)}, \end{aligned}\right\} \quad (43)$$

and

$$\bar{\bar{l}}_n^{(0)} = \frac{i\bar{\bar{k}}_n^{(0)}\{2\Omega(c_0+q)-\nu\Omega^2 R_n\}}{(c_0+q)^2+\nu\Omega^2 T_n}.$$

Critical cases may also arise when the characteristic equation has equal roots. But it can be seen by inspection that, under the conditions assigned, this cannot occur except for particular values of ν, and the matter is no further discussed.

It is to be noticed that in case (i) Ω is arbitrary. But in cases (ii) and (iii), special values of Ω are prescribed. That is, the occurrence of the special roots is only found at those positions where Ω has appropriate values.

The three special cases are now treated in detail as they arise in the terms of (33) associated with ϵ.

The terms of the variational equations associated with ϵ

Case (i). All the roots of the characteristic equation distinct.

The terms in (33) associated with the factor ϵ are

$$\left.\begin{aligned}
L(u_\lambda^{(1)}, v_\lambda^{(1)}, c_0) &= -2c_1\dot{u}_\lambda^{(0)} - 2ic_0c_1u_\lambda^{(0)} + 2\Omega c_0 v_\lambda^{(0)} + 2\Omega u_\lambda^{(0)}\dot{\sigma}_\lambda^{(1)} \\
&\quad + 2\dot{v}_\lambda^{(0)}\dot{\sigma}_\lambda^{(1)} + 2\Omega\dot{v}_\lambda^{(0)}\rho_\lambda^{(1)} - \frac{6Mu_\lambda^{(0)}\rho_\lambda^{(1)}}{a^3(\omega-\omega')^2} + 2ic_0v_\lambda^{(0)}\dot{\sigma}_\lambda^{(1)} + 2\Omega ic_0v_\lambda^{(0)}\rho_\lambda^{(1)} \\
&\quad + \Sigma\frac{u_\mu^{(0)}}{(\omega-\omega')^2}\left\{\left(\frac{\partial^3\Theta_\lambda}{\partial r_\lambda\partial r_\mu\partial r_\kappa}\right)_{00} a\rho_\kappa^{(1)} + \left(\frac{\partial^3\Theta_\lambda}{\partial r_\lambda\partial r_\mu\partial\theta_\kappa}\right)_{00}\sigma_\kappa^{(1)}\right\} \\
&\quad + \Sigma\frac{v_\mu^{(0)}}{a(\omega-\omega')^2}\left\{\left(\frac{\partial^3\Theta_\lambda}{\partial r_\lambda\partial\theta_\mu\partial r_\kappa}\right)_{00} a\rho_\kappa^{(1)} + \left(\frac{\partial^3\Theta_\lambda}{\partial r_\lambda\partial\theta_\mu\partial\theta_\kappa}\right)_{00}\sigma_\kappa^{(1)}\right\} \\
&\quad + \frac{u_\lambda^{(0)}}{(\omega-\omega')^2}\left(\frac{\partial^2\Phi_\lambda}{\partial r_\lambda^2}\right)_{00} + \frac{v_\lambda^{(0)}}{a(\omega-\omega')^2}\left(\frac{\partial^2\Phi_\lambda}{\partial r_\lambda\partial\theta_\lambda}\right)_{00}, \\
M(u_\lambda^{(1)}, v_\lambda^{(1)}, c_0) &= -2c_1\dot{v}_\lambda^{(0)} - 2ic_0c_1v_\lambda^{(0)} - 2\Omega c_1u_\lambda^{(0)} \\
&\quad - 2\rho_\lambda^{(1)}(\ddot{v}_\lambda^{(0)} + 2ic_0\dot{v}_\lambda^{(0)} - c_0^2v_\lambda^{(0)}) - 2u_\lambda^{(0)}\ddot{\sigma}_\lambda^{(1)} - 2\dot{\rho}_\lambda^{(1)}(\dot{v}_\lambda^{(0)} + ic_0v_\lambda^{(0)}) \\
&\quad - 2\Omega\dot{\rho}_\lambda^{(1)}u_\lambda^{(0)} - 2\Omega\rho_\lambda^{(1)}(\dot{u}_\lambda^{(0)} + ic_0u_\lambda^{(0)}) - 2\dot{\sigma}_\lambda^{(1)}(u_\lambda^{(0)} + ic_0u_\lambda^{(0)}) \\
&\quad + \Sigma\frac{u_\mu^{(0)}}{a(\omega-\omega')^2}\left\{\left(\frac{\partial^3\Theta_\lambda}{\partial\theta_\lambda\partial r_\mu\partial r_\kappa}\right)_{00} a\rho_\kappa^{(1)} + \left(\frac{\partial^3\Theta_\lambda}{\partial\theta_\lambda\partial r_\mu\partial\theta_\kappa}\right)_{00}\sigma_\kappa^{(1)}\right\} \\
&\quad + \Sigma\frac{v_\mu^{(0)}}{a^2(\omega-\omega')^2}\left\{\left(\frac{\partial^3\Theta_\lambda}{\partial\theta_\lambda\partial\theta_\mu\partial r_\kappa}\right)_{00} a\rho_\kappa^{(1)} + \left(\frac{\partial^3\Theta_\lambda}{\partial\theta_\lambda\partial\theta_\mu\partial\theta_\kappa}\right)_{00}\sigma_\kappa^{(1)}\right\} \\
&\quad + \frac{u_\lambda^{(0)}}{a(\omega-\omega')^2}\left(\frac{\partial^2\Phi_\lambda}{\partial\theta_\lambda\partial r_\lambda}\right)_{00} + \frac{v_\lambda^{(0)}}{a^2(\omega-\omega')^2}\left(\frac{\partial^2\Phi_\lambda}{\partial\theta_\lambda^2}\right)_{00}.
\end{aligned}\right\} \quad (44)$$

The right-hand members of these equations are all known functions of τ associated with certain arbitrary constants arising in the previous step and with the unknown parameter c_1. The parameter c_0 has already been determined from the equation (39) for the first of the three specified cases. The complementary function, obtained by solving

$$\left.\begin{aligned} L(u_\lambda^{(1)}, v_\lambda^{(1)}, c_0) &= 0, \\ M(u_\lambda^{(1)}, v_\lambda^{(1)}, c_0) &= 0, \end{aligned}\right\} \quad (45)$$

will be of the same form as (40) with new arbitrary constants. The particular integral will consist of a series of periodic terms, the only critical case arising from the constant terms of the right-hand members. Since for the selected value of c_0 equations (45) are satisfied by constant values of $u_\lambda^{(1)}$, $v_\lambda^{(1)}$, the presence of constant terms in the right-hand members of (44) will give rise to non-periodic terms. Hence for a periodic solution, c_1 must be chosen so that these constant terms are removed. To formulate the result, multiply each of equations (44) by $\frac{1}{p}e^{-2\pi is\lambda/p}$ and sum each set from $\lambda = 1$ to $\lambda = p$, putting

$$k_s^{(1)} = \frac{1}{p}\sum_{\lambda=1}^{p} u_s^{(1)}\, e^{-2\pi is\lambda/p}, \qquad l_s^{(1)} = \frac{1}{p}\sum_{\lambda=1}^{p} v_s^{(1)}\, e^{-2\pi is\lambda/p}.$$

On the right-hand side, only the constant terms will be retained. We have then

$$\left.\begin{aligned}
\bar{L}(k_s^{(1)}, l_s^{(1)}, c_0) &= -2ic_0 c_1 \bar{k}_s^{(0)} + 2\Omega c_1\, l_s^{(0)} - 6\bar{k}_s^{(0)} X_0^{(1)}\left\{\Omega^2 - \frac{K}{(\omega-\omega')^2}\right\}\\
&\quad + 2\Omega ic_0\, \bar{l}_s^{(0)} X_0^{(1)} + \sum \frac{\bar{k}_s^{(0)}\, e^{2\pi is(\mu-\lambda)/p}}{(\omega-\omega')^2}\left(\frac{\partial^3\Theta_\lambda}{\partial r_\lambda\,\partial r_\mu\,\partial r_\kappa}\right)_{00} X_0^{(1)}\\
&\quad + \sum \frac{\bar{l}_s^{(0)}\, e^{2\pi is(\mu-\lambda)/p}}{(\omega-\omega')^2}\left(\frac{\partial^3\Theta_\lambda}{\partial r_\lambda\,\partial r_\mu\,\partial \theta_\kappa}\right)_{00} X_0^{(1)} + \frac{\bar{k}_s^{(0)} B_0''}{a^3 a'(\omega-\omega')^2},\\
\bar{M}(k_s^{(1)}, l_s^{(1)}, c_0) &= 2ic_0 c_1\, l_s^{(0)} - 2\Omega c_1 \bar{k}_s^{(0)} + 2c_0^2\, \bar{l}_s^{(0)} X_0^{(1)}\\
&\quad - 2\Omega ic_0\, \bar{k}_s^{(0)} X_0^{(1)} + \sum \frac{\bar{k}_s^{(0)}\, e^{2\pi is(\mu-\lambda)}}{(\omega-\omega')^2}\left(\frac{\partial^3\Theta_\lambda}{\partial \theta_\lambda\,\partial r_\mu\,\partial r_\kappa}\right)_{00} X_0^{(1)}\\
&\quad + \sum \frac{\bar{l}_s^{(0)}\, e^{2\pi is(\mu-\lambda)}}{(\omega-\omega')^2}\left(\frac{\partial^3\Theta_\lambda}{\partial \theta_\lambda\,\partial \theta_\mu\,\partial r_\kappa}\right) X_0^{(1)}.
\end{aligned}\right\} \tag{46}$$

In order that no non-periodic terms may appear in the particular integral we must have that on multiplying the first equation of (46) by $c_0^2 + \nu\Omega^2 T_s$ and the second by $-i(2\Omega c_0 + \nu\Omega^2 Q_s)$ and adding, the right member must vanish. It appears from (38) that the ratio $l_s^{(0)}/\bar{k}_s^{(0)}$ is a pure imaginary. Further, it appears on inspection that of the terms involving derivatives of Θ_λ, those with odd θ-derivatives on summation are pure imaginaries, those with even θ-derivatives are real. It follows at once that c_1 is then a pure imaginary.

Hence when Ω is such that all the roots of the characteristic equation are distinct, the periodic solution (17) is stable to the order of the first power of ϵ.

Case (ii). The roots of the characteristic equation differ by an integer q for the same s.

We use equations (44) with the values of $u_\lambda^{(0)}$, $v_\lambda^{(0)}$ given by (42). After substitution multiply each λ-equation by $\frac{1}{p}e^{-2\pi is\lambda/p}$ and sum each set for λ from 1 to p, retaining only the critical terms, that is, those that are constants or contain the factor $e^{iq\tau}$. We then have

$$\left.\begin{aligned}
\bar{L}(k_s^{(1)}, l_s^{(1)}, c_0) = e^{iq\tau}\Big[& -2c_1 iq\bar{k}_s^{(0)} - 2ic_0 c_1 \bar{k}_s^{(0)} + 2\Omega c_1 \bar{l}_s^{(0)} \\
& + 2\Omega iq X_0^{(1)} \bar{l}_s^{(0)} - 6X_s^{(1)} \left\{\Omega - \frac{K}{(\omega-\omega')^2}\right\} \bar{k}_s^{(0)} + 2\Omega ic_0 X_0^{(1)} \bar{l}_s^{(0)} \\
& + \sum \frac{\bar{k}_s^{(0)} e^{2\pi is(\mu-\lambda)/p}}{(\omega-\omega')^2} \left(\frac{\partial^3\Theta_\lambda}{\partial r_\lambda \partial r_\mu \partial r_\kappa}\right)_{00} X_0^{(1)} \\
& + \sum \frac{\bar{l}_s^{(0)} e^{2\pi is(\mu-\lambda)/p}}{(\omega-\omega')^2} \left(\frac{\partial^3\Theta_\lambda}{\partial r_\lambda \partial r_\mu \partial \theta_\kappa}\right)_{00} X_0^{(1)} + \frac{\bar{k}_s^{(0)} B_0''}{aa'^2(\omega-\omega')^2}\Big] \\
& - 2ic_0 c_1 \bar{k}_s^{(0)} + 2\Omega c_1 \bar{l}_s^{(0)} - 6X_0^{(1)} \left\{\Omega^2 - \frac{K}{(\omega-\omega')^2}\right\} \bar{k}_s^{(0)} \\
& + \sum \frac{\bar{k}_s^{(0)} e^{2\pi i(\mu-\lambda)/p}}{(\omega-\omega')^2} \left(\frac{\partial^3\Theta_\lambda}{\partial r_\lambda \partial r_\mu \partial r_\kappa}\right)_{00} X_0^{(1)} \\
& + \sum \frac{\bar{l}_s^{(0)} e^{2\pi i(\mu-\lambda)/p}}{(\omega-\omega')^2} \left(\frac{\partial^3\Theta_\lambda}{\partial r_\lambda \partial r_\mu \partial \theta_\kappa}\right)_{00} X_0^{(1)} + \frac{\bar{k}_s^{(0)} B_0''}{aa'^2(\omega-\omega')^2}, \\
\bar{M}(k_s^{(1)}, l_s^{(1)}, c_0) = e^{iq\tau}\Big[& -2c_1 iq\bar{l}_s^{(0)} - 2ic_0 c_1 \bar{l}_s^{(0)} - 2\Omega c_1 \bar{k}_s^{(0)} \\
& + 2q^2 X_0^{(1)} \bar{l}_s^{(0)} + 4qc_0^2 X_0^{(1)} \bar{l}_s^{(0)} + 2c_0^2 X_0^{(1)} \bar{l}_s^{(0)} - 2\Omega iq X_0^{(1)} \bar{k}_s^{(0)} \\
& - 2ic_0 \Omega X_0^{(1)} \bar{k}_s^{(0)} + \sum \frac{\bar{k}_s^{(0)} e^{2\pi is(\mu-\lambda)/p}}{(\omega-\omega')^2} \left(\frac{\partial^3\Theta_\lambda}{\partial \theta_\lambda \partial r_\mu \partial r_\kappa}\right)_{00} X_0^{(1)} \\
& + \sum \frac{\bar{l}_s^{(0)} e^{2\pi is(\mu-\lambda)/p}}{a(\omega-\omega')^2} \left(\frac{\partial^3\Theta_\lambda}{\partial \theta_\lambda \partial \theta_\mu \partial r_\kappa}\right)_{00} X_0^{(1)}\Big] \\
& - 2ic_0 c_1 \bar{l}_s^{(0)} - 2\Omega c_1 \bar{k}_s^{(0)} + 2c_0^2 X_0^{(1)} \bar{l}_s^{(0)} - 2ic_0 \Omega X_0^{(1)} \bar{k}_s^{(0)} \\
& + \sum \frac{\bar{k}_s^{(0)} e^{2\pi is(\mu-\lambda)/p}}{(\omega-\omega')^2} \left(\frac{\partial^3\Theta_\lambda}{\partial \theta_\lambda \partial r_\mu \partial r_\kappa}\right)_{00} X_0^{(1)} \\
& + \sum \frac{\bar{l}_s^{(0)} e^{2\pi is(\mu-\lambda)/p}}{a(\omega-\omega')^2} \left(\frac{\partial^3\Theta_\lambda}{\partial \theta_\lambda \partial \theta_\mu \partial r_\kappa}\right)_{00} X_0^{(1)}.
\end{aligned}\right\} \quad (47)$$

The constant terms appearing in the right-hand members are of course the same as in (46). In order that no non-periodic terms should appear in the corresponding solution two conditions are to be fulfilled:

(*a*) on multiplying the first of equations (47) by $c_0^2 + \nu\Omega^2 T_s$, the second by $-i(2\Omega c_0 + \nu\Omega^2 Q_s)$ and adding, the constant terms must vanish. Since $\bar{k}_s^{(0)}/\bar{l}_s^{(0)}$ is a pure imaginary, it follows as before that we have c_1 a pure imaginary and $\bar{k}_s^{(0)}$ arbitrary;

(*b*) on multiplying the first of equations (47) by $(c_0+q)^2 + \nu\Omega^2 T_s$, the second by $-i\{2\Omega(c_0+q) + \nu\Omega^2 Q_s\}$ and adding, the sum of the terms factored by $e^{iq\tau}$ must vanish. Since $\bar{k}_s^{(0)}/\bar{l}_s^{(0)}$ is a pure imaginary it appears again that c_1 must be a pure imaginary. It will be noticed that the two sets of constants $(\bar{k}_s^{(0)}, \bar{l}_s^{(0)})$, $(\bar{k}_s^{(0)}, \bar{l}_s^{(0)})$ are independent of each other, and two values of c_1 arise from the two conditions. Hence we choose $\bar{k}_s^{(0)} = 0$ from (a) and c_1 with $\bar{\bar{k}}_s^{(0)}$ arbitrary from (b) or conversely. In either case the result is that the periodic orbits are stable at the points defined by

$$D_{c_0}^s(\Omega) = 0, \quad D_{c_0+q}^s(\Omega) = 0.$$

Case (iii). When $D_{c_0}^s(\Omega) = 0$, $D_{c_0+q}^n(\Omega) = 0$ for the same value of Ω and $s \neq n$.

Then, as stated,

$$u_\lambda^{(0)} = \bar{k}_s^{(0)} e^{2\pi is\lambda/p} + \bar{\bar{k}}_n^{(0)} e^{i(q\tau+2\pi n\lambda/p)},$$

$$v_\lambda^{(0)} = \bar{l}_s^{(0)} e^{2\pi is\lambda/p} + \bar{l}_n^{(0)} e^{i(q\tau+2\pi n\lambda/p)},$$

$$\bar{l}_s^{(0)} = \frac{i\bar{k}_s^{(0)}(2\Omega c_0 - \nu\Omega^2 R_s)}{c_2^0 + \nu\Omega^2 T_s},$$

$$\bar{l}_n^{(0)} = \frac{i\bar{\bar{k}}_n^{(0)}\{2\Omega(c_0+q) - \nu\Omega^2 R_n\}}{(c_0+q)^2 + \nu\Omega^2 T_n}.$$

On substituting in (44), critical terms arise from the constant terms in the right-hand member and from the terms involving $e^{iq\tau}$. Consider first those terms which are constants. These terms appear when $s+q = n$. Then from (44) we find, on multiplying each λ-equation by $\frac{1}{p}e^{-2\pi is\lambda/p}$ and summing the separate sets from $\lambda = 1$ to $\lambda = p$,

$$\left.\begin{aligned}
\bar{L}(k_s^{(1)}, l_s^{(1)}, c_0) = & -2ic_0c_1\bar{k}_s^{(0)} + 2\Omega c_1\bar{l}_s^{(0)} + 2\Omega qY_{-q}^{(1)}\bar{\bar{k}}_n^{(0)} + 2q^2iY_{-q}^{(1)}\bar{l}_n^{(0)} + 2\Omega iqX_{-q}^{(1)}\bar{l}_n^{(0)} \\
& -6\left\{\Omega^2 - \frac{K}{(\omega-\omega')^2}\right\}X_0^{(1)}\bar{k}_s^{(0)} - 6\left\{\Omega^2 - \frac{K}{(\omega-\omega')^2}\right\}X_{-q}^{(1)}\bar{\bar{k}}_n^{(0)} \\
& +2ic_0qY_{-q}^{(1)}\bar{l}_s^{(0)} + 2\Omega ic_0X_0^{(1)}\bar{l}_s^{(0)} + 2\Omega ic_0X_{-q}^{(0)}\bar{l}_n^{(0)} \\
& +\nu\Omega^2\{-3(N+P_s)X_0^{(1)}\bar{k}_s^{(0)} - \tfrac{3}{2}(P_s+P_n+3N+P_q)X_{-q}^{(1)}\bar{\bar{k}}_n^{(0)} \\
& \qquad +(S_n-S_s-S_q)Y_{-q}^{(1)}\bar{\bar{k}}_n^{(0)} + 4i(S_q+S_s-S_n)X_{-q}^{(1)}\bar{l}_n^{(0)} \\
& \qquad +4i(N+P_q+P_n-P_s)Y_{-q}^{(1)}\bar{l}_n^{(0)}\} \\
& +\frac{B_0''\bar{k}_s^{(0)}}{a'^3(\omega-\omega')^2} + \frac{B_{-q}''\bar{\bar{k}}_n^{(0)}}{a'^3(\omega-\omega')^2} - \frac{iqB_{-q}'\bar{l}_n^{(0)}}{aa'^2(\omega-\omega')^2}; \\
\bar{M}(k_s^{(1)}, l_s^{(1)}, c_0) = & -2ic_0c_1\bar{l}_s^{(0)} - 2\Omega c_1\bar{k}_s^{(0)} + 2q^2X_{-q}^{(1)}\bar{l}_n^{(0)} + 4c_0qX_q^{(1)}\bar{l}_n^{(0)} + 2c_0X_0^{(1)}\bar{l}_s^{(0)} \\
& +2c_0^2X_{-q}^{(1)}\bar{l}_n^{(0)} + 2q^2i\bar{\bar{k}}_n^{(0)}Y_{-q}^{(1)} - 2q^2X_{-q}^{(1)}\bar{l}_n^{(0)} - 2c_0qX_{-q}^{(1)}\bar{l}_n^{(0)} \\
& -2\Omega ic_0X_0^{(1)}\bar{k}_s^{(0)} - 2\Omega ic_0X_{-q}^{(1)}\bar{\bar{k}}_n^{(0)} - 2iq^2Y_{-q}^{(1)}\bar{\bar{k}}_n^{(0)} - 2ic_0qY_{-q}^{(1)}\bar{\bar{k}}_n^{(0)} \\
& +\nu\Omega^2\{i(S_n-S_s-S_q)X_{-q}^{(1)}\bar{\bar{k}}_n^{(0)} + i(N+P_q-P_n+P_s)Y_{-q}^{(1)}\bar{\bar{k}}_n^{(0)} \\
& \qquad +2(N+P_s)X_0^{(1)}\bar{l}_s^{(0)} + (N+P_n-P_q+P_s)X_{-q}^{(1)}\bar{l}_n^{(0)} \\
& \qquad +2(S_n-S_s-S_q)Y_{-q}^{(1)}\bar{l}_n^{(0)}\} \\
& -\frac{iqB_{-q}'\bar{\bar{k}}_n^{(0)}}{aa'^2(\omega-\omega')^2} - \frac{q^2B_{-q}\bar{l}_n^{(0)}}{a^2a'(\omega-\omega')^2}.
\end{aligned}\right\} \quad (48)$$

Next, on taking only the terms factored by $e^{iq\tau}$ in (44), multiplying each λ-equation by $\frac{1}{p}e^{-2\pi in\lambda/p}$, and summing the separate sets from $\lambda = 1$ to $\lambda = p$ with $s+q = n$, we find

$$\begin{aligned}
\overline{L}(k_s^{(1)}, l_s^{(1)}, c_0) = e^{iq\tau}\Big[& -2c_1 iq\bar{\bar{k}}_n^{(0)} - 2ic_0 c_1 \bar{\bar{k}}_n^{(0)} + 2\Omega c_1 \bar{l}_n^{(0)} - 2\Omega q Y_q^{(1)} \bar{k}_s^{(0)} \\
& + 2\Omega iq X_0^{(1)} \bar{l}_n^{(0)} - 6\left\{\Omega^2 - \frac{K}{(\omega-\omega')^2}\right\} X_q^{(1)} \bar{k}_s^{(0)} \\
& - 6\left\{\Omega^2 - \frac{K}{(\omega-\omega')^2}\right\} X_0^{(1)} \bar{\bar{k}}_n^{(0)} - 2c_0 iq Y_q^{(1)} \bar{l}_s^{(0)} \\
& \qquad + 2\Omega ic_0 X_q^{(1)} \bar{l}_s^{(0)} + 2\Omega ic_0 X_0^{(1)} \bar{l}_n^{(0)} \\
& + \nu\Omega^2\{-3(N+P_s) X_0^{(1)} \bar{\bar{k}}_n^{(0)} - \tfrac{3}{2}(P_s+P_n+3N+P_q) X_q^{(1)} \bar{k}_s^{(0)} \\
& \qquad + (S_s+S_q-S_n) Y_q^{(1)} \bar{k}_s^{(0)} + 4i(S_n-S_s-S_q) X_q^{(1)} \bar{l}_s^{(0)} \\
& \qquad + 4i(N+P_s+P_q-P_n) Y_q^{(1)} \bar{l}_s^{(0)}\} \\
& + \frac{B_0''\bar{\bar{k}}_n^{(0)}}{a'^3(\omega-\omega')^2} + \frac{B_q''\bar{k}_s^{(0)}}{a'^3(\omega-\omega')^2} + \frac{iq\bar{l}_s^{(0)} B_q'}{aa'^2(\omega-\omega')^2}\Big]; \\
\overline{M}(k_s^{(1)}, l_s^{(1)}, c_0) = e^{iq\tau}\Big[& -2iqc_1 \bar{l}_n^{(0)} - 2ic_0 c_1 \bar{l}_n^{(0)} - 2\Omega c_1 \bar{\bar{k}}_n^{(0)} + 2q^2 X_0^{(1)} \bar{l}_n^{(0)} + 4c_0 q X_0^{(1)} \bar{l}_n^{(0)} \\
& + 2c_0^2 X_0^{(1)} \bar{l}_n^{(0)} + 2c_0^2 X_q^{(1)} \bar{l}_s^{(0)} + 2iq^2 Y_q^{(1)} \bar{k}_s^{(0)} + 2c_0 q X_q^{(1)} \bar{l}_s^{(0)} \\
& - 2\Omega iq X_0^{(1)} \bar{k}_s^{(0)} - 2i\Omega q X_0^{(1)} \bar{\bar{k}}_n^{(0)} - 2i\Omega c_0 X_q^{(1)} \bar{k}_s^{(0)} \\
& - 2i\Omega c_0 X_0^{(1)} \bar{\bar{k}}_n^{(0)} + 2ic_0 q Y_q^{(1)} \bar{k}_s^{(0)} \\
& + \nu\Omega^2\{i(S_q+S_s-S_n) Y_q^{(1)} \bar{k}_s^{(0)} + i(N+P_q-P_s+P_n) Y_q^{(1)} \bar{k}_s^{(0)} \\
& \qquad + 2(N+P_n) X_0^{(1)} \bar{l}_n^{(0)} + (N+P_s-P_q+P_n) X_q^{(1)} \bar{l}_s^{(0)} \\
& \qquad + 2(S_q+S_s-S_n) Y_q^{(1)} \bar{l}_s^{(0)}\} \\
& + \frac{iqB_q'\bar{k}_s^{(0)}}{aa'^2(\omega-\omega')^2} - \frac{q^2 B_q \bar{l}_s^{(0)}}{a^2 a'(\omega-\omega')^2}\Big].
\end{aligned} \tag{49}$$

The stability in certain critical cases

In order to examine the stability of the periodic motion under the conditions of case (iii) we shall find it necessary to proceed to approximations. The criterion formulated by Maxwell for the stability of a ring of particles unperturbed by a satellite was that ν should be sufficiently small. With this condition fulfilled it is possible to evaluate approximately the quantities arising in equations (48) and (49).

The solution is dependent upon the fact that

$$D_{c_0}^s(\Omega) = 0 \quad \text{and} \quad D_{c_0+q}^n(\Omega) = 0 \tag{50}$$

for the same value of Ω; to which is added the further condition arising in (48) and (49) that $q+s = n$, q being a positive integer. The values of c_0 and Ω satisfying (50) for given values of s and n are detailed in (31). Only the results corresponding to a positive Ω are shown; those corresponding to a negative Ω refer to results applicable to rings of particles outside the orbit of the perturbing satellite and are of no physical interest.

Consider first case (31 (i)).

We have then

$$c_0/\Omega = -1+\tfrac{1}{2}\nu(N+P_n+4T_n+4Q_n),$$

$$\Omega = q\div\{2-\tfrac{1}{2}\nu(2N+P_n+P_s+4T_n+4T_s+4Q_n-4Q_s)\}.$$

Since

$$Q_n = \frac{1}{8}\sum_{m=1}^{p-1}\frac{\cos m\pi/p}{\sin^2 m\pi/p}\sin 2\pi nm/p$$

$$<\frac{1}{8}\sum_{m=1}^{p-1}\frac{\cos m\pi/p}{\sin^2 m\pi/p} = O(p^2),$$

and

$$T_n(\max) = O(p^3),$$

in retaining only the highest terms in p we may omit Q_n. Then

$$\Omega = \tfrac{1}{2}q\{1+\tfrac{7}{8}\nu(T_n+T_s)\},$$

$$c_0 = -\tfrac{1}{2}q\{1+\tfrac{7}{8}\nu(T_s-T_n)\},$$

$$D^q_q = -\tfrac{3}{4}q^4+\tfrac{1}{16}\nu q^4(7T_n+7T_s-5T_q),$$

$$\bar{l}^{(0)}_s = -2i\bar{k}^{(0)}_s(1-\nu T_s+\tfrac{7}{4}\nu T_n),$$

$$\bar{l}^{(0)}_n = 2i\bar{\bar{k}}^{(0)}_n(1-\nu T_n+\tfrac{7}{4}\nu T_s),$$

$$c_0^2+\nu\Omega^2T_s = \tfrac{1}{4}q^2(1+\tfrac{11}{4}\nu T_s-\tfrac{7}{4}\nu T_n),$$

$$(c_0+q)^2+\nu\Omega^2T_n = \tfrac{1}{4}q^2(1-\tfrac{7}{4}\nu T_s+\tfrac{11}{4}\nu T_n),$$

$$-i(2\Omega c_0+\nu\Omega^2Q_s) = \tfrac{1}{2}iq^2(1+\tfrac{7}{4}\nu T_s),$$

$$-i\{2\Omega(c_0+q)+\nu\Omega^2Q_n\} = -\tfrac{1}{2}iq^2(1+\tfrac{7}{4}\nu T_n).$$

Using these values we may reduce equations (48) and (49). They give, from (48),

$$\left.\begin{aligned}
\bar{L}(k^{(1)}_s, l^{(1)}_s, c_0) = \bar{k}^{(0)}_s\Big[&-c_1iq(1-\tfrac{9}{8}\nu T_s+\tfrac{49}{8}\nu T_n)\\
&-q^2X^{(1)}_0(\tfrac{5}{2}+\tfrac{24}{8}\nu T_s+\tfrac{35}{8}\nu T_n)+\frac{B''_0}{a'^3(\omega-\omega')^2}\Big]\\
+\bar{\bar{k}}^{(0)}_n\Big[&q^2X^{(1)}_{-q}(-\tfrac{5}{2}-\tfrac{51}{16}\nu T_n-\tfrac{67}{16}\nu T_s+\tfrac{3}{16}\nu T_q-2\nu S_q-2\nu S_s+2\nu S_n)\\
&+q^2Y^{(1)}_q(1-\tfrac{17}{8}\nu T_n+\tfrac{15}{8}\nu T_s-\nu T_q-\tfrac{1}{4}\nu S_n+\tfrac{1}{4}\nu S_s+\tfrac{1}{4}\nu S_q)\\
&+\frac{B''_q}{a'^3(\omega-\omega')^2}+\frac{2q(1-\nu T_n+\tfrac{7}{4}\nu T_s)\,B'_q}{aa'^2(\omega-\omega')^2}\Big],\\
\bar{M}(k^{(1)}_s, l^{(1)}_s, c_0) = \bar{k}^{(0)}_s&[c_1q(1-\tfrac{9}{8}\nu T_s+\tfrac{7}{8}\nu T_n)+iq^2X^{(1)}_0(-\tfrac{1}{2}+\tfrac{5}{8}\nu T_s)]\\
+\bar{\bar{k}}^{(0)}_n\Big[&iq^2X^{(1)}_q(-\tfrac{1}{2}+\tfrac{3}{4}\nu T_n-\tfrac{9}{8}\nu T_s+\tfrac{1}{4}\nu T_q-\tfrac{1}{4}\nu S_q-\tfrac{1}{4}\nu S_s+\tfrac{1}{4}\nu S_n)\\
&+iq^2Y^{(1)}_q(-1-\tfrac{3}{4}\nu T_s+\tfrac{3}{4}\nu T_n+\tfrac{1}{8}\nu T_q+\nu S_q+\nu S_s-\nu S_n)\\
&-\frac{iqB'_q}{aa'^2(\omega-\omega')^2}-\frac{2iq^2(1-\nu T_n+\tfrac{7}{4}\nu T_s)\,B_q}{a^2a'(\omega-\omega')^2}\Big].
\end{aligned}\right\}\quad(51)$$

The condition that no non-periodic terms may arise in the particular integral of these equations is obtained by multiplying the first by $\frac{1}{4}q^2(1+\frac{11}{4}\nu T_s-\frac{7}{4}\nu T_n)$ and the second by $\frac{1}{2}iq^2(1+\frac{7}{4}\nu T_s)$ and adding. The resulting value of the right-hand member should be zero.

This condition is

$$\bar{k}_s^{(0)}\Big[c_1 iq^3(\tfrac{1}{4}-\tfrac{3}{32}\nu T_s-\tfrac{21}{32}\nu T_n)+q^4X_0^{(1)}(-\tfrac{3}{8}-\tfrac{75}{32}\nu T_s)+\frac{\frac{1}{4}q^2(1+\frac{11}{4}\nu T_s-\frac{7}{4}\nu T_n)\,B_0''}{a'^3(\omega-\omega')^2}\Big]$$
$$+\bar{k}_n^{(0)}\Big[q^4X_q^{(1)}(-\tfrac{3}{8}-\tfrac{141}{64}\nu T_s-\tfrac{5}{64}\nu T_n-\tfrac{5}{64}\nu T_q-\tfrac{3}{8}\nu S_q-\tfrac{3}{8}\nu S_s+\tfrac{3}{8}\nu S_n)$$
$$+q^4Y_q^{(1)}(\tfrac{3}{4}+\tfrac{77}{32}\nu T_s-\tfrac{43}{32}\nu T_n-\tfrac{5}{16}\nu T_q+\tfrac{7}{16}\nu S_n-\tfrac{7}{16}\nu S_s-\tfrac{7}{16}\nu S_q)$$
$$+\frac{\frac{1}{4}q^2(1+\frac{11}{4}\nu T_s-\frac{7}{4}\nu T_n)\,B_q''}{a'^3(\omega-\omega')^2}+\frac{q^3(1+\frac{25}{8}\nu T_s-\frac{11}{8}\nu T_n)\,B_q'}{aa'^2(\omega-\omega')^2}$$
$$+\frac{q^4(1-\nu T_n+\frac{7}{2}\nu T_s)\,B_q}{a'a^2(\omega-\omega')^2}\Big]=0. \tag{52}$$

Similarly from (49) we have

$$\bar{L}(k_s^{(1)},l_s^{(1)},c_0)=e^{iq\tau}\Big[\bar{k}_n^{(0)}\Big\{c_1 iq(1+\tfrac{49}{8}\nu T_s-\tfrac{9}{8}\nu T_n)$$
$$-q^2X_0^{(1)}(\tfrac{5}{2}+\tfrac{24}{8}\nu T_n+\tfrac{35}{8}\nu T_s)+\frac{B_0''}{a^3(\omega-\omega')^2}\Big\}$$
$$+\bar{k}_s^{(0)}\Big\{q^2X_q^{(1)}(-\tfrac{5}{2}-\tfrac{67}{16}\nu T_n-\tfrac{51}{16}\nu T_s+\tfrac{3}{16}\nu T_q-2\nu S_q-2\nu S_s+2\nu S_n)$$
$$+q^2Y_q^{(1)}(1+\tfrac{15}{8}\nu T_n-\tfrac{17}{8}\nu T_s-\nu T_q-\tfrac{1}{4}\nu S_n+\tfrac{1}{4}\nu S_s+\tfrac{1}{4}\nu S_q)$$
$$+\frac{B_q''}{a'^3(\omega-\omega')^2}+\frac{2q(1-\nu T_s+\frac{7}{4}\nu T_n)\,B_q'}{aa'^2(\omega-\omega')^2}\Big\}\Big],$$

$$\bar{M}(k_s^{(1)},l_s^{(1)},c_0)=e^{iq\tau}\Big[\bar{k}_n^{(0)}\{c_1 q(1-\tfrac{9}{8}\nu T_n+\tfrac{7}{8}\nu T_s)+q^2iX_0^{(1)}(\tfrac{1}{2}-\tfrac{5}{8}\nu T_n)\}$$
$$+\bar{k}_s^{(0)}\Big\{iq^2X_q^{(1)}(\tfrac{1}{2}-\tfrac{3}{4}\nu T_s+\tfrac{9}{8}\nu T_n-\tfrac{1}{4}\nu T_q+\tfrac{1}{4}\nu S_q+\tfrac{1}{4}\nu S_s-\tfrac{1}{4}\nu S_n)$$
$$+iq^2Y_q^{(1)}(1-\tfrac{3}{4}\nu T_s+\tfrac{3}{4}\nu T_n-\tfrac{1}{8}\nu T_q-\nu S_q-\nu S_s+\nu S_n)$$
$$+\frac{iqB_q'}{aa'^2(\omega-\omega')^2}+\frac{2iq^2(1-\nu T_s+\frac{7}{4}\nu T_n)\,B_q}{a^2a'(\omega-\omega')^2}\Big\}\Big]. \tag{53}$$

The condition that no non-periodic terms may arise in the particular integral of these equations is obtained by multiplying the first by $\frac{1}{4}q^2(1-\frac{7}{4}\nu T_s+\frac{11}{4}\nu T_n)$, the second

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by $-\frac{1}{2}iq^2(1+\frac{7}{4}\nu T_n)$ and adding. The resulting value of the right-hand member must be zero. That is,

$$\bar{\bar{k}}_n^{(0)}\left\{c_1 iq^3(-\tfrac{1}{4}+\tfrac{21}{32}\nu T_s+\tfrac{3}{32}\nu T_n)+q^4X_0^{(1)}(-\tfrac{3}{8}-\tfrac{75}{32}\nu T_n)+\frac{\frac{1}{4}q^2(1-\frac{7}{4}\nu T_s+\frac{11}{4}\nu T_n)}{a'^3(\omega-\omega')^2}\right\}$$
$$+\bar{k}_s^{(0)}\left\{q^4X_q^{(1)}(-\tfrac{3}{8}-\tfrac{5}{64}\nu T_s-\tfrac{141}{64}\nu T_n-\tfrac{5}{64}\nu T_q-\tfrac{3}{8}\nu S_q-\tfrac{3}{8}\nu S_s+\tfrac{3}{8}\nu S_n)\right.$$
$$+q^4Y_q^{(1)}(\tfrac{3}{4}-\tfrac{43}{32}\nu T_s+\tfrac{77}{32}\nu T_n-\tfrac{5}{16}\nu T_q+\tfrac{7}{16}\nu S_n-\tfrac{7}{16}\nu S_s-\tfrac{7}{16}\nu S_q)$$
$$+\frac{\frac{1}{4}q^2(1-\frac{7}{4}\nu T_s+\frac{11}{4}\nu T_n)\,B''_q}{a'^3(\omega-\omega')^2}+\frac{q^3(1-\frac{11}{8}\nu T_s+\frac{25}{8}\nu T_n)\,B'_q}{aa'^2(\omega-\omega')^2}$$
$$\left.+\frac{q^4(1-\nu T_s+\frac{7}{2}\nu T_n)\,B_q}{a'a^2(\omega-\omega')^2}\right\}=0. \tag{54}$$

The eliminant of (52) and (54) gives a quadratic equation for c_1, upon the character of the roots of which depend the stability or instability. Without a knowledge of p it is not possible to work out exactly the index c_1 but certain inferences can be made. For small integers q, T_{s+q} and T_s differ by a quantity of lower order than either. With sufficient accuracy therefore we may put $T_n = T_s$ provided $(n-s)$ is not a large integer. In addition T_q for a small integer q may be neglected in comparison with T_s. Also the maximum value of $(S_n-S_s-S_q)$ is of the same order as the maximum value of T_s. We shall write it $\bar{S}_s$. Equations (52) and (54) may therefore be written

$$\left.\begin{aligned}
&\bar{k}_s^{(0)}\left\{c_1 iq^3(\tfrac{1}{4}-\tfrac{3}{4}\nu T_s)+q^4X_0^{(1)}(-\tfrac{3}{8}-\tfrac{75}{32}\nu T_s)+\frac{\frac{1}{4}q^2(1+\nu T_s)\,B''_0}{a'^3(\omega-\omega')^2}\right\}\\
&\quad+\bar{\bar{k}}_n^{(0)}\left\{q^4X_q^{(1)}(-\tfrac{3}{8}-\tfrac{73}{32}\nu T_s-\tfrac{3}{8}\nu\bar{S}_s)+q^4Y_q^{(1)}(\tfrac{3}{4}+\tfrac{17}{16}\nu T_s-\tfrac{7}{16}\nu\bar{S}_s)\right.\\
&\quad\left.+\frac{\frac{1}{4}q^2(1+\nu T_s)\,B''_q}{a'^3(\omega-\omega')^2}+\frac{q^3(1+\frac{7}{4}\nu T_s)\,B'_q}{aa'^2(\omega-\omega')^2}+\frac{q^4(1+\frac{5}{2}\nu T_s)}{a'a^2(\omega-\omega')^2}\right\}=0,\\
&\bar{\bar{k}}_n^{(0)}\left\{c_1 iq^3(-\tfrac{1}{4}+\tfrac{3}{4}\nu T_s)+q^4X_0^{(1)}(-\tfrac{3}{8}-\tfrac{75}{32}\nu T_s)+\frac{\frac{1}{4}q^2B''_0(1+\nu T_s)}{a'^3(\omega-\omega')^2}\right\}\\
&\quad+\bar{k}_s^{(0)}\left\{q^4X_q^{(1)}(-\tfrac{3}{8}-\tfrac{73}{32}\nu T_s-\tfrac{3}{8}\nu\bar{S}_s)+q^4Y_q^{(1)}(\tfrac{3}{4}+\tfrac{17}{16}\nu T_s-\tfrac{7}{16}\nu\bar{S}_s)\right.\\
&\quad\left.+\frac{\frac{1}{4}q^2(1+\nu T_s)\,B''_q}{a'^3(\omega-\omega')^2}+\frac{q^3(1+\frac{7}{4}\nu T_s)\,B'_q}{aa'^2(\omega-\omega')^2}+\frac{q^4B_q(1+\frac{5}{2}\nu T_s)}{a'a^2(\omega-\omega')^2}\right\}=0.
\end{aligned}\right\} \tag{55}$$

Whence

$$c_1^2q^2(\tfrac{1}{4}-\tfrac{3}{4}\nu T_s)^2+\left\{\frac{\frac{1}{4}(1+\nu T_s)\,B''_0}{a'^3(\omega-\omega')^2}-q^2(\tfrac{3}{8}+\tfrac{75}{32}\nu T_s)\,X_0^{(1)}\right\}^2$$
$$=\left\{-q^2X_q^{(1)}(\tfrac{3}{8}-\tfrac{73}{32}\nu T_s+\tfrac{3}{8}\nu\bar{S}_s)+q^2Y_q^{(1)}(\tfrac{3}{4}+\tfrac{17}{16}\nu T_s-\tfrac{7}{16}\nu\bar{S}_s)\right.$$
$$\left.+\frac{\frac{1}{4}(1+\nu T_s)\,B''_q}{a^3(\omega-\omega')^2}+\frac{q^3(1+\frac{7}{4}\nu T_s)\,B'_q}{aa'^2(\omega-\omega')^2}+\frac{q^4(1+\frac{5}{2}\nu T_s)\,B_q}{a'a^2(\omega-\omega')^2}\right\}^2. \tag{56}$$

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The nature of the stability then depends upon the relative magnitudes of the two terms independent of c_1. It is known from Maxwell's theory that for the stability of the ring unperturbed by a satellite the condition to be fulfilled is $\nu T_s < 0{\cdot}039$. We may use νT_s as a parameter varying with s between the limits 0 and a value $\not> 0{\cdot}039$, and hence

$$\tfrac{1}{2}q \leqslant \Omega \leqslant \frac{q}{2-\frac{7}{2}\nu T_s}.$$

Equation (56) will determine the stability between these limits.

Consider first the case where $\Omega = \frac{1}{2}q$; that is, $T_s = \overline{S}_s = 0$ for some s. Evaluating the coefficients $X_q^{(1)}$, $Y_q^{(1)}$, we find

$$\frac{\frac{1}{4}B_0''}{a'^3(\omega-\omega')^2} - \tfrac{3}{8}q^2 X_0^{(1)} = \frac{B_0' + \frac{1}{2}\alpha B_0''}{2aa'^2(\omega-\omega')^2}; \tag{57}$$

and $$-\tfrac{3}{8}q^2 X_q^{(1)} + \tfrac{3}{4}q^2 Y_q^{(1)} + \frac{\frac{1}{4}B_q''}{a'^3(\omega-\omega')^2} + \frac{qB_q'}{aa'^2(\omega-\omega')^2} + \frac{q^2 B_q}{a^2 a'(\omega-\omega')^2}$$

$$= \frac{1}{2aa'^2(\omega-\omega')^2}\{\tfrac{1}{2}\alpha^2 B_q'' + \alpha B_q'(2q-\tfrac{1}{2}) + B_q(2q^2-\tfrac{5}{4}q)\}. \tag{58}$$

Omitting the denominators of (57) and (58), we have the following numerical results:

q	Ω	α $(=a/a')$	(57)	(58)
2	1	0	0	0
3	$\frac{3}{2}$	0·48	0·282	2·61
4	2	0·63	0·775	7·23
5	$\frac{5}{2}$	0·71	1·427	14·42
6	3	0·76	2·24	21·95

It is clear that the periodic orbits at $\Omega = \frac{1}{2}q$, for $q > 2$, are unstable. They will also be unstable in the vicinity of $\Omega = \frac{1}{2}q$, but detailed calculation in each case would be necessary to determine the range of values of Ω in the vicinity of this point. The margin of instability shown by the table in the cases of $q = 4$ and higher is clearly such as to ensure that the orbits are unstable over the whole range of values of Ω given by the inequality $\frac{1}{2}q \leqslant \Omega \leqslant \frac{\frac{1}{2}q}{1-\frac{7}{4}\nu T_s}$. But it is doubtful whether this range will be covered in the cases $q = 2, 3$.

Consider next the solution (31 (iv)). This gives, on retaining only the highest order terms,

$$\Omega = \frac{q}{\nu^{\frac{1}{2}}\{\sqrt{(3T_n)}+\sqrt{(3T_s)}\}},$$

$$c_0 = -\nu^{\frac{1}{2}}\sqrt{(3T_s)}\,\Omega,$$

$$D_q^q = \frac{q^4}{\nu\{\sqrt{(3T_n)}+\sqrt{(3T_s)}\}^2},$$

$$X_q^{(1)} = \frac{q^2\Omega B_q}{a^2a'(\omega-\omega')^2 D_q^q}, \quad q \neq 0,$$

$$X_0^{(1)} = -\frac{B_0'}{3aa'^2(\omega-\omega')^2\Omega^2},$$

$$Y_q^{(1)} = \frac{3qB_q\Omega^2}{2a^2a'(\omega-\omega')^2 D_q^q},$$

$$c_0^2+\nu\Omega^2T_s = 4\nu\Omega^2T_s,$$

$$(c_0+q)^2+\nu\Omega^2T_n = \frac{4T_nq^2}{\{\sqrt{(3T_n)}+\sqrt{(3T_s)}\}^2},$$

$$-i(2\Omega c_0+\nu\Omega^2Q_s) = 2i\nu^{\frac{1}{2}}\Omega^2\sqrt{(3T_s)},$$

$$-i\{2\Omega(c_0+q)+\nu\Omega^2Q_s\} = -\frac{2iq^2\sqrt{(3T_n)}}{\nu^{\frac{1}{2}}\{\sqrt{(3T_s)}+\sqrt{(3T_n)}\}^2},$$

$$\bar{l}_s^{(0)} = \frac{-i\sqrt{3}\bar{k}_s^{(0)}}{2\nu^{\frac{1}{2}}\sqrt{(T_s)}},$$

$$\bar{\bar{l}}_n^{(0)} = \frac{i\bar{\bar{k}}_n^{(0)}\,2\sqrt{(3T_n)}}{(T_s+3T_n)\,\nu^{\frac{1}{2}}}. \tag{59}$$

Substituting these values in (48) and retaining only the terms of highest order,

$$\bar{L}(k_s^{(1)}, l_s^{(1)}, c_0) = -\frac{c_1 i\Omega\sqrt{3}\bar{k}_s^{(0)}}{\sqrt{(\nu T_s)}} + \text{terms in } \nu^{-\frac{1}{2}} \text{ and smaller;}$$

$$\bar{M}(k_s^{(1)}, l_s^{(1)}, c_0) = c_1\Omega\bar{k}_s^{(0)} + \frac{4\sqrt{(3\nu T_n)}\,i\Omega^2Y_{-q}^{(1)}(S_n-S_s-S_q)\,\bar{\bar{k}}_n^{(0)}}{T_s+3T_n} - \frac{2\sqrt{(3T_n)}\,q^2iB_{-q}\bar{\bar{k}}_n^{(0)}}{a^2a'(\omega-\omega')^2(T_s+3T_n)\,\nu^{\frac{1}{2}}}.$$

On multiplying these by $4\nu\Omega^2T_s$ and $2i\nu^{\frac{1}{2}}\Omega^2\sqrt{(3T_s)}$ respectively and adding, the condition that no non-periodic terms shall appear in the particular integral of these equations is

$$-2c_1 i\Omega^3\sqrt{(3\nu T_s)}\,\bar{k}_s^{(0)} + \frac{24\nu\Omega^4\sqrt{(T_sT_n)}\,Y_{-q}^{(1)}(S_s+S_q-S_n)\,\bar{\bar{k}}_n^{(0)}}{T_s+3T_n} + \frac{12\sqrt{(T_sT_n)}\,q^2\Omega^2B_{-q}\bar{\bar{k}}_n^{(0)}}{a^2a'(\omega-\omega')^2(T_s+3T_n)} = 0. \tag{60}$$

Similarly from equations (49) we find

$$\bar{L}(k_s^{(1)}, l_s^{(1)}, c_0) = e^{iq\tau}\left[\frac{4c_1 i\Omega\sqrt{(3T_n)}\,\bar{\bar{k}}_n^{(0)}}{(T_s+3T_n)\,\nu^{\frac{1}{2}}} + \text{terms in } \nu^{-\frac{1}{2}} \text{ and smaller}\right];$$

$$\bar{M}(k_s^{(1)}, l_s^{(1)}, c_0) = e^{iq\tau}\left[\frac{-2c_1 q(T_s-3T_n)}{\{\sqrt{(3T_s)}+\sqrt{(3T_n)}\}(T_s+3T_n)\,\nu^{\frac{1}{2}}} + \frac{\nu\Omega^2 i\sqrt{3}Y_q^{(1)}(S_n-S_s-S_q)\,\bar{k}_s^{(0)}}{\sqrt{(\nu T_s)}} + \frac{iq^2\sqrt{3}B_q\,\bar{k}_s^{(0)}}{2a^2a'(\omega-\omega')^2\sqrt{(\nu T_s)}}\right].$$

On multiplying by $\dfrac{4q^2T_n}{\{\sqrt{(3T_s)}+\sqrt{(3T_n)}\}^2}$ and $-\dfrac{2iq^2\sqrt{(3T_n)}}{\nu^{\frac{1}{2}}\{\sqrt{(3T_s)}+\sqrt{(3T_n)}\}^2}$ respectively and adding, the condition that no non-periodic terms shall appear in the particular integral is

$$\frac{4c_1 iq\sqrt{3}(T_s+T_n)\,\bar{k}_n^{(0)}}{\nu(T_s+3T_n)\{\sqrt{(3T_s)}+\sqrt{(3T_n)}\}^3} + \frac{6\Omega^2(S_n-S_s-S_q)\,Y_q^{(1)}\bar{k}_s^{(0)}}{\sqrt{(T_s)}} + \frac{3q^2B_q\,\bar{k}_s^{(0)}}{\nu a^2a'(\omega-\omega')^2\sqrt{(T_s)}} = 0. \quad (61)$$

On eliminating the ratio $\bar{k}_n^{(0)}/\bar{k}_s^{(0)}$ from (60) and (61) and reducing, we have

$$2T_s(T_n+T_s)\,\nu c_1^2 = 3q\sqrt{(T_sT_n)}[3\nu(S_n-S_s-S_q)+q\nu\{\sqrt{(3T_s)}+\sqrt{(3T_n)}\}^2]^2\frac{B_q^2}{a^4a'^2(\omega-\omega')^4}. \quad (62)$$

Hence the periodic solution is unstable at points defined by

$$\Omega = \frac{q}{\nu^{\frac{1}{2}}\{\sqrt{(3T_s)}+\sqrt{(3T_n)}\}}.$$

As already pointed out, T_{s+q} and T_s differ, for small values of the integer q, by a quantity of lower order than either. Hence this formula may be written, with sufficient accuracy,

$$\Omega = \frac{q}{2\sqrt{(3\nu T_s)}}.$$

Passing to the case (31 (v)) we have, retaining only the principal terms,

$$\Omega = \frac{q}{1-\sqrt{(3\nu T_s)}}, \quad c_0 = q\sqrt{(3\nu T_s)},$$

$$X_q^{(1)} = \frac{aB_q'+2qa'B_q}{4a^2a'^2(\omega-\omega')^2q^2\sqrt{(3\nu T_s)}}, \quad q \neq 0,$$

$$X_0^{(1)} = -\frac{B_0'}{3q^2aa'^2(\omega-\omega')^2},$$

$$Y_q^{(1)} = 2X_q^{(1)},$$

$$c_0^2+\nu\Omega^2T_s = 4\nu T_s q^2,$$

$$(c_0+q)^2+\nu\Omega^2 T_n = q^2,$$
$$-i(2\Omega c_0+\nu\Omega^2 Q_s) = -2iq^2\surd(3\nu T_s),$$
$$-i\{2\Omega(c_0+q)+\nu\Omega^2 Q_s\} = -2i\Omega q,$$
$$\bar{l}_s^{(0)} = \frac{i\surd 3\bar{k}_s^{(0)}}{2\surd(\nu T_s)},$$
$$\bar{l}_n^{(0)} = 2i\bar{k}_n^{(0)}.$$

Using those values in equations (48) and retaining only the highest order of terms, we find

$$\left.\begin{aligned}\bar{L}(k_s^{(1)}, l_s^{(1)}, c_0) &= -\frac{c_1 iq\surd 3\bar{k}_s^{(0)}}{\surd(\nu T_s)}+2\Omega q\, Y_{-q}^{(1)}\,\bar{k}_n^{(0)}\\ &\quad -4q^2 Y_{-q}^{(1)}\,\bar{k}_n^{(0)}-4q^2 X_{-q}^{(1)}\,\bar{k}_n^{(0)}-6q^2 X_{-q}^{(1)}\,\bar{k}_n^{(0)};\\ \bar{M}(k_s^{(1)}, l_s^{(1)}, c_0) &= qc_1\,\bar{k}_s^{(0)}+2iq^2\surd(3\nu T_s)\,X_{-q}^{(1)}\,\bar{k}_n^{(0)}\\ &\quad -2iq^2\surd(3\nu T_s)\,Y_{-q}\bar{k}_n^{(0)}.\end{aligned}\right\} \tag{63}$$

In order that no non-periodic terms may appear in the particular integral of equations (63) the sum of the product of the right-hand members of the first and $4\nu q^2 T_s$, and the product of the right-hand member of the second and $-2iq^2\surd(3\nu T_s)$ must be zero. That is

$$6c_1 iq^3\surd(3\nu T_s)\,\bar{k}_s^{(0)}+\frac{\bar{k}_n^{(0)}\surd(\nu T_s)}{\surd 3a^2a'^2(\omega-\omega')^2}\left(\tfrac{3}{2}q^2 aB'_q+3q^3 a'B_q\right) = 0. \tag{64}$$

Similarly, equations (49) give

$$\left.\begin{aligned}\bar{L}(k_s^{(1)}, l_s^{(1)}, c_0) &= e^{iq\tau}\left[2c_1 iq\bar{k}_n^{(0)}+q^2(Y_q^{(1)}-9X_q^{(1)})\,\bar{k}_s^{(0)}-\frac{q\surd 3B'_q\bar{k}_s^{(0)}}{2aa'^2(\omega-\omega')^2\surd(\nu T_s)}\right],\\ \bar{M}(k_s^{(1)}, l_s^{(1)}, c_0) &= e^{iq\tau}\left[2c_1 q\bar{k}_n^{(0)}+q^2 i(Y_q^{(1)}+X_q^{(1)})\,\bar{k}_s^{(0)}-\frac{q^2 i\surd 3B_q\bar{k}_s^{(0)}}{2a^2a'(\omega-\omega')^2\surd(\nu T_s)}\right].\end{aligned}\right\} \tag{65}$$

In order that no non-periodic terms may arise in the particular integral of (65), the sum of the product of the right-hand member of the first and q^2, and the product of the right-hand member of the second and $-2iq^2$ should be zero. That is

$$-2c_1 iq^3\bar{k}_n^{(0)}+\frac{\bar{k}_s^{(0)}}{4a^2a'^2(\omega-\omega')^2\surd(3\nu T_s)}\{(3q^2-6q^3)\,aB'_q+(6q^3-6q^4)\,a'B_q\} = 0, \tag{66}$$

On eliminating $\bar{k}_n^{(0)}/\bar{k}_s^{(0)}$ we find

$$12c_1^2 q^2 = \frac{\{\tfrac{3}{2}aB'_q+3qa'B_q\}\{(1-2q)\,aB'_q+2(q-q^2)\,a'B_q\}}{4a^4a'^4(\omega-\omega')^4}. \tag{67}$$

Hence, since $q \geqslant 1$, c_1 is a pure imaginary. The periodic solution is then stable at the positions given by

$$\Omega = q\{1-\surd(3\nu T_s)\}^{-1}.$$

The conjugate value $\Omega = q\{1+\surd(3\nu T_s)\}^{-1}$ similarly shows stability.

The further cases (31 (ii)), (31 (iii)) are involved in those already considered.

For completeness, we should proceed to the equations associated with higher powers of ϵ. Owing to the complexity and extent of the expressions this is hardly possible with satisfaction. The inclusion of such terms cannot change the zones of instability already obtained though it might lead to additional zones of instability.

General consideration of the results

It has been shown that a system consisting of a large primary, surrounded by a ring of numerous small particles perturbed by a co-planar satellite, possesses a family of periodic orbits for which the parameter is Ω. For certain values of Ω, the periodic orbits do not exist. In general the remaining orbits are stable; but for values of Ω defined by

$$\Omega = \frac{\frac{1}{2}q}{1-\frac{7}{4}\nu T_s}, \tag{68}$$

and

$$\Omega = \frac{q}{2\surd(3\nu T_s)}, \tag{69}$$

the orbits are unstable. By using νT_s as a variable in s, with a lower limit zero, and a finite, though small, upper limit, the formulae (67), (68) are seen to define certain ranges of the parameter Ω.

For $q = 1$, (68) gives negative values for the rotation of the ring. For $q = 2, 3$ there is some doubt as to the range of instability, but there is certainly instability at $\Omega = \frac{3}{2}$. For higher values of q the instability is clear. The values of Ω for differing integers q and the varying values of νT_s are therefore seen to determine a series of zones of instability. These zones have their inner edges at $\Omega = \frac{1}{2}q$ and extend outwards in each case. Further, the zones do not overlap so long as $q+1 < \dfrac{4}{7\nu T_s}$.

In applying (69) we need only consider the lowest value of q for which instability occurs. This gives a single zone extending from a point defined by the maximum value of T_s up to the perturbing satellite. The zones for higher values of q lie within the last. This formula thus prescribes an outer limit beyond which no stable ring can exist.

The points at which no periodic orbits exist as given by (27) and (28) are readily seen to lie within one or other of these zones.

It should be noted that the whole of this theory is based upon periodic orbits of the particles forming the ring. The production of periodic orbits, however, requires special initial conditions, and these conditions may not exist. But, as pointed out by Moulton (1914), the characteristics of the motion of a non-periodic orbit are analogous to those of a periodic orbit in the same general neighbourhood. That is, if a periodic orbit is decidedly stable, a neighbouring non-periodic orbit, produced by slightly varying the initial conditions, is also stable, and conversely. Hence the results found for periodic orbits may be regarded as of general application.

Application to the Saturnian system

In proceeding to apply the preceding results to the ring system of Saturn, it is necessary to make a choice of satellite. The innermost, Mimas, might be considered most appropriate in affecting the ring. But the second satellite, Enceladus, has four times the mass and is only about $\frac{1}{3}$ further away from Saturn. Also Tethys, while still further, is again much greater in mass. We shall therefore consider each satellite independently.

The quantity νT_s appearing in the formulae is unknown except that according to Maxwell it must have a certain small upper limit. Its maximum value will be determined from the known dimensions of the Cassini division of the ring.

The dimensions of the ring as observed by Lowell (1916) are:

Outer radius of ring A	20·01″,
Inner radius of ring A	17·64″,
Outer radius of ring B	16·87″,
Inner radius of ring B	13·00″,
Radius of Encke's division	19·00″.

Consider each satellite in turn.

I. *Mimas*. $a' = 26{\cdot}8''$.

The inner boundary of each zone as given by (68) is $\Omega = \frac{1}{2}q$. Omitting $q = 1, 2$ for reasons already stated, we have

$q = 3$,	$\Omega = \frac{3}{2}$,	$\alpha = 0{\cdot}481$,	$a = 12{\cdot}89''$
4	2	0·630	16·87″
5	$\frac{5}{2}$	0·711	19·05″
6	3	0·763	20·44″

Each of these corresponds to a marked feature of Saturn's ring. The first is near the inner edge of ring B; the second is at the inner edge of Cassini's division; the third is at the position of Encke's division; and the fourth is just outside the boundary of ring A.

The maximum value of νT_s for the system can be determined from the measured radius of the outer boundary of Cassini's division. Improved accuracy may be obtained by using the complete equation (22) in place of the approximation (68). Using $a = 17{\cdot}64''$ from the table and $a' = 26{\cdot}8''$, we find that νT_s has the maximum value 0·0342.

With this value of νT_s, formula (68) will give the outer boundary of each zone. For $q = 3$, the consequent zone would penetrate into ring B. But, as pointed out earlier, it is doubtful whether the range of instability at $q = 3$ is so wide, though it certainly exists at the point $\Omega = \frac{3}{2}$.

Next use the determined value of νT_s in (69), or, better, use the original equation (22). We should then have for $q = 1$ a zone of instability extending from $a = 12{\cdot}54''$ to the orbit of Mimas, thus blotting out the whole ring. It is readily shown, however, that the right-hand member of (62) for $q = 1$ is exceedingly small. Compare the values for $q = 1$ and $q = 2$ of the principal part of (62), viz.

$$\surd(q)\,[3\nu(S_n - S_s - S_q) + q\nu\{\surd(3T_s) + \surd(3T_n)\}^2]\,\frac{B_q}{a^2 a'(\omega - \omega')^2}, \tag{70}$$

at the points where νT_s has its maximum value. It appears that for $q = 1$, (70) has the value 0·00088, and for $q = 2$, it has the value 0·217. The modulus of instability at $q = 1$ thus is small compared with that at $q = 2$; so small that, for a satisfactory result, it would be necessary to include in the computation the terms of lower order which were neglected. We may take it that if the orbits are unstable at $q = 1$, the modulus of instability is very small.

On taking $q = 2$, the zone of instability commences at $20{\cdot}2''$ and extends outwards to the orbit of Mimas. This result defines the outer edge of ring A with fair accuracy.

Lastly, applying (68) to the case $q = 5$, and using the value of νT_s just found, the outer edge of this zone is determined at $19{\cdot}56''$. This corresponds to the observations of the Encke division, but would seem to give a width rather greater than the observed gap, which is, however, difficult to measure.

For higher values of q the divisions are obliterated by the large zone arising from (69).

The largest value of νT_s permitted by the Maxwell theory of a simple ring without a satellite is 0·039. The value derived by the present work from the dimensions of the Cassini division is 0·0342, just below the Maxwell limit.

II. *Enceladus*. $a' = 34{\cdot}4''$.

Omit the values $q = 1, 2$ as before.

$q = 3,$	$\Omega = \frac{3}{2},$	$\alpha = 0{\cdot}481,$	$a' = 16{\cdot}54''$
3	2	0·630	$21{\cdot}6''$

The division beginning at $16{\cdot}54''$ is within ring B. There is no observed division at this point. That at $21{\cdot}6''$ is outside the observed ring system.

None of the remaining satellites give divisions in the observed ring system. Each could, however, give a zone of instability corresponding to $q = 2$ near to the origin (really in the vicinity of the surface of Saturn). But as pointed out, it is doubtful from the analysis whether this zone of instability really exists, and it is therefore not surprising to find that the whole ring system is not dissipated by the large mass of Titan.

The results of this analysis are to show that the observed ring system of Saturn is due to the innermost satellite Mimas. The inner edge of ring B and the outer edge of ring A are indicated with considerable accuracy. The Cassini division and the Encke division are also accounted for. The masses of the particles (taken as equal in this work and therefore as representing the mean value of those in the actual ring) are just below the limiting value prescribed by the Maxwell criterion.

References

Brown, E. W. 1924 *Proc. Nat. Acad. Sci., Wash.*, **10**, 248.
Goldsbrough, G. R. 1921 *Phil. Trans.* A, **222**, 101.
Greaves, W. M. H. 1922 *Mon. Not. R. Astr. Soc.* **82**, 356.
Lowell, P. 1916 *Lowell Observatory Bulletin*, no. 68.
Moulton, F. R. 1914 *Mon. Not. R. Astr. Soc.* **75**, 40.
Moulton and others 1920 *Periodic Orbits*, XIV. Washington.
Pendse, C. A. 1935 *Phil. Trans.* A, **234**, 735.
Poincaré, H. 1892 *Les Méthodes Nouvelles de la Méc. Céleste*, I.
Poincaré, H. 1900 *Figures d'équilibre d'une masse fluide*.